Instabilities of shear layers

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Preliminary version

Abstract

This article gathers notes of two lectures given at Grenoble's University in June 2023, and is an introduction to recent works on shear layers, in collaboration with Y. Guo, T. Nguyen and B. Pausader.

Updated versions will be posted on Arxiv.

Contents

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1 Introduction

In this paper, we are interested in the inviscid limit of the incompressible Navier Stokes equations with Dirichlet boundary condition in the half plane. The classical incompressible Navier Stokes equations in the half space $\Omega = \{y > 0\}$ read

$$\partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla p^\nu - \nu \Delta u^\nu = 0, \tag{1.1}$$

$$\nabla \cdot u^\nu = 0, \tag{1.2}$$

$$u^\nu = 0 \quad \text{when} \quad y = 0. \tag{1.3}$$

As ν goes to 0, formally, Navier Stokes equations degenerate into the incompressible Euler equations

$$\partial_t u + (u \cdot \nabla) u + \nabla p = 0, \tag{1.4}$$

$$\nabla \cdot u = 0, \tag{1.5}$$

$$u \cdot n = 0 \quad \text{when} \quad y = 0, \tag{1.6}$$

where n is a vector normal to the boundary.

Despite many efforts, the question of the convergence of solutions of Navier Stokes equations to solutions of Euler equations is widely open. A classical approach is to introduce the so called Prandtl's boundary layer, of size $O(\sqrt{\nu})$. This leads to look for u^ν under the form

$$u^\nu(t,x,y) = u^{Euler}(t,x,y) + u^{Prandtl}(t,x,\nu^{-1/2} y) + O(\sqrt{\nu})_{L^\infty}, \tag{1.7}$$

where u^{Euler} solves Euler equations and $u^{Prandtl}$ is a corrector, located near the boundary, solution of the so called Prandtl's equations (see [12] for instance). Such an expansion is however far from being fully justified. Roughly speaking, the attempts to answer this question follow four different directions

- Proof of (1.7) in small time for initial data with analytic type regularity. This direction has been pioneered by R.E. Caflish and M. Sammartino [27, 28] and has later been extended to data with Gevrey regularity, or to initial data with vorticity supported away from the boundary. These results in particular show that the formal analysis of Prandtl is valid in the particular case of analytic data [11, 24]. In that direction, note also the construction of stationary solutions for Blasius' boundary layer [17],[25]. We in particular refer the readers to the works of N. Masmoudi, D. Gérard-Varet, Y. Maekawa, Y. Guo, V. Vicol.

- Proof that (1.7) does not hold because Prandtl's equation has singularities: blow up [9], boundary separation [8].

- Proof that (1.7) is false because of underlying linear instabilities in the case of solutions with Sobolev regularity. These instabilities lead to the successive onset of various sublayers, of size $\nu^{1/2}$ as expected, but also $\nu^{3/4}$ or $\nu^{13/16}$ as will be detailed in this article. This includes the proof of the linear instability of shear flows [16], the instability of Prandtl layers in L^∞ for a particular class of initial data [15] and the instability of Blasius boundary layer under Sobolev perturbations [13].

- Kato's result, which is a singular and isolated result [20]: u^ν converges to u in $L^2([0,T]\times\Omega)$ if and only if

$$\nu \int_0^T \int_{d(x,\partial\Omega)\le\nu} \|\nabla u^\nu\|^2 \, dx\, dt \to 0$$

as ν goes to 0. Namely, solutions of Navier Stokes equations converge to a solution of Euler equation if the energy dissipation in a layer of size $O(\nu)$ goes to 0. A striking feature of Kato's result is that the size which appears is ν and not $\sqrt{\nu}$, the classical size of Prandtl boundary layer. Therefore the question of convergence of solutions of Navier Stokes equations is linked to the capability of these solutions to create boundary layers of size ν, starting from a classical Prandtl layer of size $\sqrt{\nu}$.

In a paradoxical way, the first two approaches manage to justify classical results from Physics, namely the importance of Prandtl boundary layer [27, 28], or a classical scenario of boundary separation [8], but at the expense of the analyticity of the solution, or its time independence.

The third and fourth approaches are also paradoxical, since, whereas the requirements on the smoothness of the initial data are natural (H^s or L^2 regularity), the boundary layers which appear in these studies are unphysical, like $\nu^{13/16}$ or ν. They do not appear in the classical physics literature and seem mere mathematical artefacts.

The aim of these lectures is to discuss the third approach and to discuss this paradox. The plan is the following. In section 2 we introduce the Rayleigh and Orr Sommerfeld equations, which are studied in more details in section 3 and 4, where we obtain optimal bounds on the solutions to linearised Euler and Navier Stokes equations. Sections 5 and 6 are devoted to the study of two different kinds of instabilities of shear flows. In section 7 we discuss multilayers instabilities, and then give perspectives in section 8.

Before jumping into the details we state the two main results that are discussed in these lectures. We will consider particular solutions of the Navier Stokes equations called "shear layers", namely solutions $V_0^\nu(t,y)$ of the form

$$V_0^\nu(t,y) = \begin{pmatrix} U_s(t,\nu^{-1/2}y) \\ 0 \end{pmatrix}, \tag{1.8}$$

where U_s is a given smooth function. For such flows, Navier Stokes equations reduce to the heat equation

$$\partial_t U_s - \partial_{yy} U_s = 0, \tag{1.9}$$

together with the boundary condition

$$U_s(t,0) = 0. \tag{1.10}$$

Note in particular that V_0^ν is analytic for any time, provided $U_s(0,\cdot)$ is analytic in y. We define

$$U_+ = \lim_{y\to\infty} U_s(0,y),$$

and will assume that $U_+ \neq 0$.

It turns out that non trivial shear layers are always linearly unstable provided the viscosity is small enough [16]. The nature of the instability depends on whether the profile is stable for linearised Euler equations or not (a notion detailed below). This leads to the following two different results:

Theorem 1.1. *Let $U_s(0,\cdot)$ be a smooth profile which is spectrally unstable for Euler equations. Then for any N and s arbitrarily large, there exists a sequence of solutions V^ν of Navier Stokes equations with forcing term F^ν and a sequence of times T^ν such that*

$$\|V^\nu(0,\cdot,\cdot) - V_0^\nu(0,\cdot)\|_{H^s} \le \nu^N, \tag{1.11}$$

$$\|F^\nu\|_{L^\infty([0,T^\nu],H^s)} \le \nu^N, \tag{1.12}$$

$$\|V^\nu(T^\nu,\cdot,\cdot) - V_0^\nu(T^\nu,\cdot)\|_{L^\infty} \ge \sigma > 0, \tag{1.13}$$

$$T^\nu \sim C_0 \log \nu^{-1}, \tag{1.14}$$

and such that, as $t \to T^\nu$, $V^\nu(1\cdot,\cdot)$ exhibits the scales $\nu^{1/2}$, $\nu^{3/4}$ and $\nu^{13/16}$ in the y variable.

If we start with a profile U_s which is spectrally stable for Euler equations, for instance a monotonic profile, the instability process is different and we have

Theorem 1.2. *Let $U_s(0,\cdot)$ be a smooth profile which is spectrally stable for Euler equations. Then for any N and s arbitrarily large, there exists a sequence of solutions V^ν of Navier Stokes equations with forcing term F^ν, and a sequence of times T^ν such that*

$$\|V^\nu(0,\cdot,\cdot) - V_0^\nu(0,\cdot)\|_{H^s} \le \nu^N, \tag{1.15}$$

$$\|F^\nu\|_{L^\infty([0,T^\nu],H^s)} \le \nu^N, \tag{1.16}$$

$$\|V^\nu(T^\nu,\cdot,\cdot) - V_0^\nu(T^\nu,\cdot)\|_{L^\infty} \ge \sigma\nu^{1/4} > 0, \tag{1.17}$$

$$T^\nu \sim C_0 \log \nu^{-1}, \tag{1.18}$$

and such that, as $t \to T^\nu$, u^ν exhibits the scales, $\nu^{1/2}$ and $\nu^{5/8}$ in the y variable.

These Theorems are discussed in sections 5, 6 and 7. For the proofs, we refer to [16] and [5].

2 Fourier Laplace transform

2.1 Warm up

Let us first recall Laplace's method. Let A be a matrix, and let $\phi(t)$ be the solution of

$$\frac{d\phi}{dt}(t) = A\phi(t),$$

with initial data $\phi(0) = \phi_0$. Then we have the following representation formula

$$\phi(t) = e^{tA}\phi_0 = \frac{1}{2i\pi}\int_\Gamma e^{\lambda t}(A-\lambda)^{-1}\phi_0 \, d\lambda, \tag{2.1}$$

where Γ is an integration contour "on the right" of the spectrum of A. Using analyticity we can then play with this contour, and shift it to the left till it reaches the point spectrum of A. We can further shift it to the left, going through the point spectrum of A, provided we add the residues at these singularities. According to Cauchy's formula, this leads to

$$\phi(t) = \sum_{\lambda \in Sp(A)} P_\lambda(\phi_0)e^{\lambda t} + \frac{1}{2i\pi}\int_\Gamma (A-\lambda)^{-1}\phi_0 \, d\lambda$$

where $Sp(A)$ is the spectrum of A, P_λ is the projection on the eigenspace corresponding to the eigenvalue λ (assuming that the eigenvalue is simple to simplify the presentation), and where the contour Γ is now on the left of the spectrum. Note that the corresponding integral is now exponentially decaying, with an arbitrary fast speed.

In these notes we will follow exactly the same strategy with A replaced by linearised Euler or Navier Stokes operators.

2.2 Orr Sommerfeld and Rayleigh equations

In this section we introduce the classical Rayleigh and Orr Sommerfeld equations. Let L_{NS} be the linearised Navier Stokes operator near a shear layer profile $U_s(y)$, namely

$$L_{NS}v = \Big((U_s(y),0)\cdot\nabla\Big)v + \Big(v\cdot\nabla\Big)(U_s(y),0) - \nu\Delta v + \nabla q, \tag{2.2}$$

with $\nabla\cdot v = 0$ and Dirichlet boundary condition. Let $v(t)$ be a solution of

$$\partial_t v(t) + L_{NS}v(t) = 0$$

with initial data $v(0) = v_0$. Then, as for (2.1), we have the representation formula

$$v(t) = \frac{1}{2i\pi}\int_\Gamma e^{\lambda t}\,(L_{NS}-\lambda)^{-1}v_0\,d\lambda, \tag{2.3}$$

where Γ is any contour on the right of the spectrum of L_{NS}. We thus need to study the resolvent of L_{NS}, namely to study the equation

$$(L_{NS}-\lambda)u = v_0. \tag{2.4}$$

We take advantage of the incompressibility condition and introduce the stream function $\psi(t,x,y)$ of $u(t,x,y)$ in such a way that

$$u = \nabla^\perp\psi.$$

We further take the Fourier transform $\psi_\alpha(t,y)$ of $\psi(t,x,y)$ in the horizontal variable x, with dual Fourier variable $\alpha\in\mathbb{R}$. Following the tradition [12], we define c by

$$\lambda = -i\alpha c.$$

This leads to write the horizontal Fourier transform $u_\alpha(t,y)$ of $u(t,x,y)$ under the form

$$u_\alpha = \nabla^\perp\Big(e^{i\alpha x}\psi_\alpha(y)\Big) = e^{i\alpha x}(\partial_y\psi_\alpha, -i\alpha\psi_\alpha).$$

Note that the horizontal Fourier transform ω_α of the vorticity ω is simply given by

$$\omega_\alpha = -(\partial_y^2-\alpha^2)\psi_\alpha. \tag{2.5}$$

Taking the curl of (2.4) we then get

$$OS_{\alpha,c,\nu}(\psi) := (U_s-c)(\partial_y^2-\alpha^2)\psi - U_s''\psi - \frac{\nu}{i\alpha}(\partial_y^2-\alpha^2)^2\psi = -\frac{\omega_{0,\alpha}}{i\alpha}, \tag{2.6}$$

where

$$\omega_{0,\alpha} = (\nabla\times v_0)_\alpha$$

denotes the horizontal Fourier transform of the curl of the initial data v_0. The Dirichlet boundary condition gives

$$\psi_\alpha(0) = \partial_y \psi_\alpha(0) = 0. \tag{2.7}$$

Moreover, $\psi_\alpha(y) \to 0$ as $y \to +\infty$. We also introduce

$$\varepsilon = \frac{\nu}{i\alpha}. \tag{2.8}$$

As ν goes to 0, the Orr Sommerfeld operator $OS_{\alpha,c,\nu}$ degenerates into the classical Rayleigh operator

$$Ray_{\alpha,c}(\psi) = (U_s - c)(\partial_y^2 - \alpha^2)\psi - U_s''\psi \tag{2.9}$$

which is a second order operator, together with the boundary condition

$$\psi(0) = 0. \tag{2.10}$$

The next two sections are devoted to a detailed study of Rayleigh (section 3) and Orr Sommerfeld (section 4), which allow to obtain optimal bounds on the solutions to linearised Euler and Navier Stokes equations thanks to (2.3).

3 Rayleigh equation

3.1 Link with linearised Euler equations

Note that $Ray_{\alpha,c}^{-1}$ is the resolvent of the linearised Euler operator, in stream function formulation and after an horizontal Fourier transform. Namely, let L_E denote the linearised Euler operator

$$L_E u = \Big((U_s(y), 0) \cdot \nabla\Big) u + \Big(u \cdot \nabla\Big)(U_s(y), 0) + \nabla p,$$

and let $u(t)$ be the solution of

$$\partial_t u = L_E u$$

with initial data u^0. Let ψ_α^0 be the Fourier transform of the stream function of the initial data u^0, and, with a slight abuse of notation, let us denote by $e^{L_E t}\psi_\alpha^0$ the stream function of $u(t)$. Then, using $\lambda = -i\alpha c$, we have

$$e^{L_E t}\psi_\alpha^0 = -\frac{1}{2i\pi}\int_\Gamma e^{-i\alpha c t} Ray_{\alpha,c}^{-1}\omega_{0,\alpha}\, dc \tag{3.1}$$

where Γ is a contour "above" the spectrum of $Ray_{\alpha,c}$ (as $\lambda = -i\alpha c$, the contour is now "above" the spectrum).

We note that (3.1) gives an explicit expression of the solution of linearised Euler equations, in terms of the inverse of Rayleigh operator. As we will see, a detailed analysis of the Rayleigh operator allows to give optimal bounds on the behavior of solutions of linearised Euler equations, and in particular to investigate the so called "inviscid damping".

The first step is to study the spectrum of linearised Euler equations. The spectrum σ_{Euler} of L_E is composed of a point spectrum σ_P and of a continuous spectrum σ_C. We note that $Ray_{\alpha,c}$ degenerates when $U_s(y) = c$, hence the continuous spectrum σ_C is simply the range of U_s. Now, as U_s has an exponential behavior, for c out of the range of U_s, there exists only one solution $\psi_{\alpha,c}(y)$ of (2.9) such that $\psi_{\alpha,c}(y) \sim e^{-|\alpha| y}$ as $y \to +\infty$. We thus have

$$\sigma_{Euler} = \sigma_P \cup \sigma_C, \qquad \sigma_P = \Big\{ c \quad | \quad \psi_{\alpha,c}(0) = 0 \Big\}, \qquad \sigma_C = Range(U_s).$$

As $\psi_{\alpha,c}(0)$ is an holomorphic function of c outside the range of U_s, the eigenvalues are isolated. It is even possible to prove that they are in finite number.

Let us recall Rayleigh's criterium:

Proposition 3.1. *If U_s has no inflection point, then $\sigma_P \cap \{\Im c > 0\} = \emptyset$.*

Proof. Let ψ and c be an eigenvector and eigenvalue of Rayleigh operator. Then

$$(\partial_y^2 - \alpha^2)\psi = \frac{U_s''}{U_s - c}\psi.$$

We multiply by $\bar{\psi}$, which gives

$$-\int |\partial_y \psi|^2 + \alpha^2 |\psi|^2 \, dy = \int \frac{U_s''}{U_s - c} |\psi|^2 \, dy.$$

We then take the imaginary part, which gives

$$\Im c \int U_s'' \frac{|\psi|^2}{|U_s - c|^2} \, dy = 0.$$

Thus U_s'' must change sign, hence U_s must have an inflection point. □

In particular, monotonic profiles are spectrally stable (i.e., $\sigma_P \cap \{\Im c > 0\} = \emptyset$). Rayleigh's criterium has been refined by Fjortoft (see [12]). Up to now, there is no necessary and sufficient condition for a profile U_s to be spectrally stable. In particular some profiles with inflection points are stable.

3.2 Construction of two particular solutions of Rayleigh equation

For the simplicity of the presentation we restrict ourselves to profiles U_s which are monotonic, with $U_s'(y) > 0$. If c is away from the range of U_s, then $U_s(y) - c$ does not vanish. Rayleigh equation is then a regular ordinary differential equation. We thus focus on c close to the range of U_s, where Rayleigh equation degenerates. For c close enough to $Range(U_s)$, there exists some complex number y_c such that

$$U_s(y_c) = c.$$

As $U_s(y)$ is monotonic in y, for a given real c, there exists at most one such y_c, which simplifies the analysis. Such a complex number y_c is called a "critical layer". We note that Rayleigh operator $Ray_{\alpha,c}$ degenerates at this critical layer. A first solution of (2.9) is obtained by looking for an holomorphic solution which vanishes at y_c. Looking for this first solution $\psi^r_{\alpha,c}(y)$ of the form

$$\psi^r_{\alpha,c}(y) = \sum_{n \geq 1} \beta_n(\alpha, c)(y - y_c)^n,$$

it is easy to get a recurrence relation on the coefficients $\beta_n(\alpha, c)$, depending on the Taylor expansion of U_s near y_c, and to prove that the corresponding series has a positive radius of convergence. Then $\psi^r_{\alpha,c}$ may be extended to a "regular" solution on the whole real line.

An independent solution is obtained through the method of the variation of constant, and is explicitly given by

$$\psi^s_{\alpha,c}(y) = \psi^r_{\alpha,c}(y) \int_{y^\star}^{y} \frac{dz}{\psi^r_{\alpha,c}(z)^2},$$

where $y^\star$ can be chosen arbitrarily. Expanding $\psi^r_{\alpha,c}$ near y_c in this expression, we observe that, near y_c, this second solution behaves like

$$\psi^s_{\alpha,c}(y) = P^s_{\alpha,c}(y) + Q^s_{\alpha,c}(y)\,(y-y_c)\,\log(y-y_c)$$

where $P^s_{\alpha,c}$ and $Q^s_{\alpha,c}$ are holomorphic functions near y_c. It is important to note that while $\psi^r_{\alpha,c}$ is smooth on the real line, $\psi^s_{\alpha,c}$ is non holomorphic at y_c, with a "$(y-y_c)\log(y-y_c)$" singularity. This is a "small singularity" for the stream function, however the corresponding vorticity behaves like

$$\omega^s_{\alpha,c}(y) \sim \frac{Q^s_{\alpha,c}(y_c)}{y-y_c}$$

and is thus singular at y_c. Of course all the solutions of $Ray_{\alpha,c}\ \psi = 0$ are a linear combination of $\psi^r_{\alpha,c}$ and $\psi^s_{\alpha,c}$. Physically, at y_c there is a "resonance" between the speed of the flow $U_s(y_c)$ and the wave, leading to this singularity on the vorticity.

Now, using linear combinations of $\psi^r_{\alpha,c}$ and $\psi^s_{\alpha,c}$, it is possible to construct $\psi_{\pm,\alpha}(y,c)$ such that

$$\psi_{\pm,\alpha}(y,c) \sim e^{\pm|\alpha|y}$$

as $y \to +\infty$, with unit Wronskian. We note that both $\psi_{\pm,\alpha}(y,c)$ may be singular at y_c and are of the form

$$\psi_{\pm,\alpha}(y,c) = P_{\pm,\alpha}(y-y_c,c) + (y-y_c)\log(y-y_c)Q_{\pm,\alpha}(y-y_c,c) \tag{3.2}$$

where $P_{\pm,\alpha}$ and $Q_{\pm,\alpha}$ are holomorphic. Of course $\psi_{-,\alpha}(y,c)$ goes to 0 as $y \to +\infty$, while $\psi_{+,\alpha}(y,c)$ diverges at infinity. We note that $\psi_{\pm,\alpha}(y,c)$ have "logarithmic branches" at $y = y_c$.

3.3 Green function for Rayleigh equation

Using $\psi_{\pm,\alpha}(y,c)$, it is possible to explicitly construct the Green function of $Ray_{\alpha,c}$, defined by

$$Ray_{\alpha,c}\Big(G_{\alpha,c}(x,y)\Big) = \delta_x \tag{3.3}$$

and by $G_{\alpha,c}(x,0) = 0$ and $G_{\alpha,c}(x,y) \to 0$ as $y \to +\infty$. For this we first solve (3.3) without the boundary conditions, just choosing a particular solution. This leads to the introduction of

$$G^{int}_{\alpha,c}(x,y) = \frac{1}{U_s(x)-c}\begin{cases} \psi_{-,\alpha}(y,c)\psi_{+,\alpha}(x,c) & \text{if} \quad y > x \\ \psi_{-,\alpha}(x,c)\psi_{+,\alpha}(y,c) & \text{if} \quad y < x, \end{cases}$$

keeping in mind that the Wronskian between $\psi_{+,\alpha}$ and $\psi_{-,\alpha}$ is one. We then correct $G^{int}_{\alpha,c}$ by $G^b_{\alpha,c}$ such that

$$G_{\alpha,c} = G^{int}_{\alpha,c} + G^b_{\alpha,c}$$

satisfies both Rayleigh's equation and the two boundary conditions. This leads to

$$G^b_{\alpha,c}(x,y) = -\psi_{+,\alpha}(0,c)\frac{\psi_{-,\alpha}(y,c)}{\psi_{-,\alpha}(0,c)}\frac{\psi_{-,\alpha}(x,c)}{U_s(x)-c}$$

We note that $G_{\alpha,c}$ is singular at $c = U_s(y)$ and $c = U_s(x)$, with poles and "logarithmic branches" at this points.

The solution of Rayleigh's equation

$$(U_s - c)(\partial_y^2 - \alpha^2)\psi - U_s''\psi = f, \tag{3.4}$$

together with its two boundary conditions, is then explicitly given by

$$\psi(y) = \int_0^{+\infty} G_{\alpha,c}(x,y)f(x)\,dx. \tag{3.5}$$

3.4 Inviscid damping

We now combine (3.1) and (3.5) to study the asymptotic behavior of solutions to Euler equations. Bounds on solutions of linearised Euler equations have been pioneered in physics by F. Bouchet and H. Morita in [7], and later rigorously proved in [1, 18, 19, 32, 33, 34]. The main feature is that solutions of linearised Euler equations decay like t^{-1} if there is no unstable eigenvalue, a phenomena known as "inviscid damping".

In this section, following the lines of [2], we prove the inviscid damping in a simplified case by combining (3.1) and (3.5). To simplify the presentation, we assume that $U_s(y)$ is holomorphic in y, satisfies $U_s(0) = 0$, and is monotonically increasing in y, with $U_s'(y) > 0$.

Theorem 3.2. *Assume that $U_s(y)$ is holomorphic in y, satisfies $U_s(0) = 0$, and is monotonically increasing in y. Let $\langle t \rangle = 1 + t$. Assume that the initial data $\omega_0(x)$ is a smooth function, then*

$$\|\psi_\alpha(t,\cdot)\|_{L^\infty} + \|\partial_y \psi_\alpha(t,\cdot)\|_{L^\infty} + \|u_\alpha(t,\cdot)\|_{L^\infty} \lesssim \langle t \rangle^{-1}. \tag{3.6}$$

Moreover there exists $\omega_\infty(t,y)$ such that

$$\omega_\alpha(t,y) = \omega_\infty(t,y) e^{-i\alpha U_s(y) t} + O(\langle t \rangle^{-1}). \tag{3.7}$$

Note that, as U_s is monotonically increasing $\sigma_P \cap \{\Im c > 0\} = \emptyset$. Hence there is no unstable mode. The first step is to move Γ close to σ_C in (3.1) and to choose for instance

$$\Gamma = \{(1+i)\mathbb{R}_- + m + i\varepsilon\} \cup \{[m,M] + i\varepsilon\} \cup \{(1-i)\mathbb{R}_+ + M + i\varepsilon\}$$

for some small ε. However, with this choice of Γ we only get an exponential bound, of the form $e^{\varepsilon t}$. To get the $\langle t \rangle^{-1}$ decay, we need to go "through" σ_C, and hence to study how Rayleigh operator behaves close to *Range*(U_s). Let

$$\psi_{\alpha,c} = -(i\alpha)^{-1}\, Ray_{\alpha,c}^{-1}\, \omega_{0,\alpha}.$$

Then $\psi_{\alpha,c}$ is explicitly given using formula (3.5), and we have

$$e^{L_E t} \psi_\alpha^0 = \frac{\alpha}{2\pi} \int_\Gamma e^{-i\alpha c t} \psi_{\alpha,c}(y)\, dc.$$

We further observe that the corresponding vorticity

$$\omega_{\alpha,c} = (\partial_y^2 - \alpha^2)\psi_{\alpha,c}$$

is directly given by Rayleigh's equation

$$\omega_{\alpha,c} = \frac{i\alpha U_s'' \psi_{\alpha,c} + \omega_{\alpha,0}}{\alpha(U_s - c)}. \tag{3.8}$$

Thus the horizontal Fourier transform $\omega_\alpha(t)$ of the solution $u(t)$ is given by

$$\omega_\alpha(t,y) = \frac{\alpha}{2\pi} \int_\Gamma e^{-i\alpha c t} \frac{\zeta_\alpha(y,c)}{U_s(y) - c}\, dc \tag{3.9}$$

where

$$\zeta_{\alpha,c} = i\alpha U_s'' \psi_{\alpha,c} + \omega_{\alpha,0}. \tag{3.10}$$

We will first discuss the following result

Proposition 3.3. *Under the assumptions of Theorem 3.2, as $t \to +\infty$,*

$$\omega_\alpha(t,y) = \zeta_\alpha(y) e^{-i\alpha U_s(y) t} + O(t^{-1}). \tag{3.11}$$

Proof. We split ζ_α according to (3.10). Assuming that $\omega_{\alpha,0}$ is holomorphic, we can move Γ downwards to $\Im c < 0$, and, thanks to Cauchy's residue theorem, the contribution of the pole at $c = U_s(y)$ gives

$$\frac{\alpha}{2\pi} \int_\Gamma e^{-i\alpha c t} \frac{\omega_{\alpha,0}(y)}{U_s(y) - c} \, dc = e^{-i\alpha U_s(y) t} \omega_{\alpha,0}(y) + O(e^{-\varepsilon t})$$

for some positive ε. The other term is

$$I := \frac{\alpha}{2\pi} \int_\Gamma e^{-i\alpha c t} U_s''(y) \frac{\psi_\alpha(y,c)}{U_s(y) - c} \, dc. \tag{3.12}$$

We then insert

$$\psi_\alpha(y,c) = \int G_\alpha(x,y) \psi_\alpha^0(x) \, dx$$

in (3.12). This leads to various terms, and amongst them

$$\int_{x<y} \int_\Gamma U_s''(y) \frac{\psi_{+,\alpha}(x,c)}{U_s(x) - c} \frac{\psi_{-,\alpha}(y,c)}{U_s(y) - c} \, dc \, dx.$$

Let us fix x and y. We have to study

$$I := \int_\Gamma \frac{\psi_{+,\alpha}(x,c)}{U_s(x) - c} \frac{\psi_{-,\alpha}(y,c)}{U_s(y) - c} \, dc$$

We note that the integrand is singular when $c = U_s(y)$ or $c = U_s(x)$, with two types of singularities: poles and "logarithmic branches". When we shift Γ downwards, the poles give contributions through Cauchy's residue theorem. The "logarithmic branch" lead to consider terms of the form

$$J := \int_\Gamma R(c) \log(y - y_c)$$

where $R(c)$ is holomorphic near y_c. We choose the determination of the logarithmic so that $c \to \log(y - y_c)$ is well defined except on a vertical half line. To bound J we shift Γ downwards, making the "turn" of the singularity through two nearby vertical lines, which gives a t^{-1} contribution. The polynomial decay thus arises from the logarithmic branch created by Rayleigh's singularity. □

We now go back to Theorem 3.2. To go from the vorticity to the stream function we need to invert the Laplace operator $\partial_y^2 - \alpha^2$. Let $H_\alpha(x,y)$ be the Green function of $(\partial_y^2 - \alpha^2)$ with Dirichlet boundary condition. We have

$$H_\alpha(x,y) = \frac{e^{-|\alpha||x-y|}}{2|\alpha|} - \frac{e^{-|\alpha||x+y|}}{2|\alpha|}.$$

Then

$$\psi_\alpha(x) = \int_0^{+\infty} H_\alpha(x,y) \omega_\alpha(t,y) \, dx = \int_0^{+\infty} H_\alpha(x,y) \zeta_\alpha(y) e^{-i\alpha U_s(y) t} \, dx + O(t^{-1}) \tag{3.13}$$

and the end of the computations are straightforward.

4 Orr Sommerfeld equation

Let us now study the Orr Sommerfeld operator. We refer to [16] for detailed proofs. We only qualitatively describe the construction of solutions to Orr Sommerfeld equation in this section, without details and give hints on the long time behavior of solutions to linearised Navier Stokes equations (see [5] for more details).

4.1 Construction of four independent solutions

"Slow" and "fast" solutions

Let us first consider the Orr Sommerfeld operator alone, without any boundary condition. It is a fourth order differential operator, therefore we have to construct four independent solutions to completely describe its solutions. Let us first study their behavior at infinity. At infinity OS "degenerates" into

$$(U_+ - c)(\partial_y^2 - \alpha^2)\psi - \varepsilon(\partial_y^2 - \alpha^2)^2\psi = 0$$

or equivalently

$$\Big[(U_+ - c) - \varepsilon(\partial_y^2 - \alpha^2)\Big](\partial_y^2 - \alpha^2)\psi = 0,$$

hence, as $y \to +\infty$, either $\psi \sim e^{\pm|\alpha|y}$ or $\psi \sim e^{\pm\sigma y}$ where

$$\sigma = \sqrt{\frac{U_+ - c}{\varepsilon}}.$$

In the sequel, c will be small, hence σ will be of order $|\varepsilon^{-1/2}| \gg 1$. It can thus be proven that there exists four independent solutions to this operator, two with a "slow" behavior (like $e^{\pm|\alpha|y}$), called $\psi_{s,\pm}$, and two with a "fast" behavior (like $e^{\pm\sigma y}$), called $\psi_{f,\pm}$. The "-" subscript refers to solutions which go to 0 at infinity and the "+" subscript to solutions diverging as y goes to $+\infty$.

"Fast" solutions

"Fast" solutions have very large derivatives. For these solutions, Orr Sommerfeld may be approximated by its higher order derivatives, namely by

$$(U_s - c)\partial_y^2 - \varepsilon\partial_z^4 = Airy\,\partial_y^2, \tag{4.1}$$

where $Airy$ is the modified Airy operator

$$Airy = (U_s - c) - \varepsilon\partial_y^2.$$

More precisely,

$$OS = Airy\,\partial_y^2 + Rem,$$

where it turns out that the remainder operator Rem may be treated as a perturbative term.

The first step is thus to construct solutions of $Airy\ \partial_y^2 = 0$, or, up to two integrations in y, to construct solutions of $Airy = 0$. Near y_c, the $Airy$ operator may further be approximated by the classical Airy operator

$$U_s'(y_c)(y - y_c) - \varepsilon\partial_y^2. \tag{4.2}$$

Solutions of (4.2) may be explicitly expressed as combinations of the classical Airy's functions Ai and Bi, which are solutions of the classical Airy equation

$$y\psi = \partial_y^2\psi.$$

More precisely, $Ai(\gamma(y-y_c))$ and $Bi(\gamma(y-y_c))$ are solutions of (4.2) provided

$$\gamma = \Big(\frac{i\alpha U_s'(y_c)}{\nu}\Big)^{1/3}. \tag{4.3}$$

Now, solutions of $Airy = 0$ may be constructed, starting from solutions of (4.2), through the so called Langer's transformation (which is a transformation of the phase and amplitude of the solution). To go back to (4.1) we then have to integrate twice these solutions. As a consequence, fast solutions to the Orr Sommerfeld equation may be expressed in terms of second primitives of the classical Airy functions Ai and Bi. Let us call $Ai(2,.)$ and $Bi(2,.)$ this second primitives of Ai and Bi. The second step is to take into account the remainder term Rem, through an iterative scheme. We skip the technical details.

The output of this construction is a complete description of $\psi_{\pm,f}$. In particular, at leading order,

$$\psi_{-,f}(y) = Ai\Big(2, \gamma(y-y_c)\Big) + O(\alpha).$$

Moreover, it can be proven that $\psi_{f,-}$ satisfies

$$\psi_{f,-}(0) = Ai(2, -\gamma y_c) + O(\alpha) \tag{4.4}$$

and

$$\partial_y \psi_{f,-}(0) = \gamma Ai(1, -\gamma y_c) + O(1). \tag{4.5}$$

"Slow" solutions

For the "slow" solutions, as a first approximation, higher order derivatives may be neglected. We thus see OS as a perturbation of Rayleigh's operator as ν goes to 0, namely

$$OS = Ray - \varepsilon Diff,$$

where

$$Diff(\psi) = (\partial_z^2 - \alpha^2)^2 \psi.$$

For "slow" solutions, $Diff$ may be treated as a small perturbation. This is indeed true for $\psi^r_{\alpha,c}$, which is smooth near y_c, and we have

$$OS_{\alpha,c,\nu}(\psi^r_{\alpha,c}) = O(\varepsilon).$$

However, for $\psi^s_{\alpha,c}$ the situation is more delicate since $\varepsilon \partial_y^4 \psi^s_{\alpha,c}$ behaves like $\varepsilon(y-y_c)^{-3}$ near the critical layer. As a consequence, away from the critical layer, $\psi^s_{\alpha,c}$ is a "good" approximate solution to OS. However, close to the critical layer, the viscous term $\varepsilon Diff(\psi^s_{\alpha,c})$ can no longer be neglected. A "boundary layer" at y_c, more exactly an "inner layer", must be added at y_c to "hide" the $\varepsilon(y-y_c)^{-3}$ singularity. We thus correct $\psi^s_{\alpha,c}$ by adding ψ^c, solution to

$$Airy\, \partial_y^2 \psi^c = \varepsilon Diff(\psi^s_{\alpha,c}).$$

Note that ψ^c may be explicitly computed using Airy functions. It turns out that $\psi^s_{\alpha,c} + \psi^c$ is then a "good enough" solution to the Orr Sommerfeld equation. It is then possible to construction two independent solutions to OS (see [16]), and to combine them in order to have one which goes to 0 at infinity, called $\psi_{s,-}$. Further computations [12] lead to

$$\psi_{s,-}(0) = -c + \alpha \frac{U_+^2}{U_s'(0)} + O(\alpha^2) \tag{4.6}$$

and

$$\partial_y \psi_{s,-}(0) = U_s'(0) + O(\alpha). \tag{4.7}$$

4.2 Dispersion relation and related instability

An eigenmode of Orr Sommerfeld equation (together with its boundary conditions) is a combination of these four particular solutions which goes to 0 at infinity and which vanishes, together with its first derivative, at $y = 0$. As an eigenmode must go to 0 as y goes to infinity, it is a linear combination of $\psi_{f,-}$ and $\psi_{s,-}$ only. To match the boundary conditions at $y = 0$ there must exist nonzero constants a and b such that

$$a\psi_{f,-}(0) + b\psi_{s,-}(0) = 0$$

and

$$a\partial_y\psi_{f,-}(0) + b\partial_y\psi_{s,-}(0) = 0.$$

The dispersion relation is therefore

$$\frac{\psi_{f,-}(0)}{\partial_y\psi_{f,-}(0)} = \frac{\psi_{s,-}(0)}{\partial_y\psi_{s,-}(0)} \tag{4.8}$$

or, using the previous computations,

$$\alpha\frac{U_+^2}{U_s'(0)^2} - \frac{c}{U_s'(0)} = \gamma^{-1}\frac{Ai(2,-\gamma y_c)}{Ai(1,-\gamma y_c)} + O(\alpha^2). \tag{4.9}$$

To simplify the discussion we will focus on the particular case where α and c are both of order $\nu^{1/4}$. It turns out that this is an area where instabilities occur, and we conjecture that it is in this region that the most unstable instabilities may be found. We rescale α and c by $\nu^{1/4}$ and introduce

$$\alpha = \alpha_0\nu^{1/4}, \quad c = c_0\nu^{1/4}, \quad Z = \gamma y_c,$$

which leads to

$$\alpha_0\frac{U_+^2}{U_s'(0)^2} - \frac{c_0}{U_s'(0)} = \frac{1}{(i\alpha_0U_s'(y_c))^{1/3}}\frac{Ai(2,-Z)}{Ai(1,-Z)} + O(\nu^{1/4}). \tag{4.10}$$

Note that as $U_s(y_c) = c$,

$$y_c = U_s'(0)^{-1}c + O(c)$$

and

$$Z = \Big(iU_s'(y_c)\Big)^{1/3}\alpha_0^{1/3}U_s'(0)^{-1}c_0 + O(\nu^{1/4}).$$

Note that the argument of Z equals $\pi/6$. We then introduce the following function, called Tietjens function, of the real variable z

$$Ti(z) = \frac{Ai(2, ze^{-5i\pi/6})}{ze^{-5i\pi/6}Ai(1, ze^{-5i\pi/6})}.$$

At first order the dispersion relation becomes

$$\alpha_0\frac{U_+^2}{U_s'(0)} = c_0\Big[1 - Ti(-Ze^{5i\pi/6})\Big]. \tag{4.11}$$

This dispersion relation can be numerically investigated. Note that it only depends on the limit U_+ of the horizontal velocity at infinity and on $U_s'(0)$. It can be proven that, provided α_0 is large enough, there exists c_0 with $\Im c_0 > 0$, leading to an unstable mode.

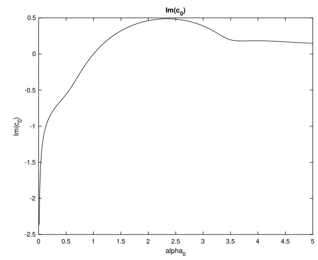

Figure 1: $\Im c_0$ as a function of α_0

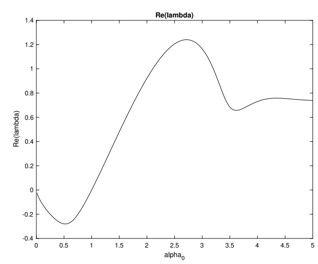

Figure 2: $\Re\lambda$ as a function of α_0

Let us numerically illustrate this dispersion relation in the case $U_s'(0) = U_+ = 1$. Figure (a) shows the imaginary part of c_0 as a function of α_0. We see that there exists a constant α_c such that $\Im c_0 > 0$ if $\alpha_0 > \alpha_c$ and $\Im c_0 < 0$ if $\alpha_0 < \alpha_c$. Figure (b) shows $\Re\lambda = \alpha_0 \Im c_0$ as a function of α_0. We see that there exists a unique global maximum to this function, at $\alpha_0 = \alpha_M \sim 2.8$.

Let us now detail the corresponding linear instability. Its stream function ψ_{lin} is of the form

$$\psi_{lin} = b\psi_{s,-} + a\psi_{f,-}.$$

Choosing $b = 1$ we see that $a = O(\nu^{1/4})$, hence

$$\psi_{lin}(y) = U_s(y) - c + \alpha \frac{U_+^2}{U_s'(0)} + aAi\Big(2, \gamma(y - y_c)\Big) + O(\nu^{1/2}). \tag{4.12}$$

The corresponding horizontal and vertical velocities u_{lin} and v_{lin} are given by

$$u_{lin} = \partial_y \psi_{lin} = U_s'(y) + \gamma a Ai\Big(1, \gamma(y - y_c)\Big) + O(\nu^{1/4}), \tag{4.13}$$

and

$$v_{lin} = -i\alpha\psi_{lin} = O(\nu^{1/4}). \tag{4.14}$$

Note that $\gamma a = O(1)$. The first term in (4.13) may be seen as a "displacement velocity", corresponding to a translation of U_s. The second term is of order $O(1)$ and located in the boundary layer, namely within a distance $O(\nu^{1/4})$ to the boundary. Note that the vorticity

$$\omega_{lin} = -(\partial_y^2 - \alpha^2)\psi_{lin}(y) = -U_s''(y) - \gamma^2 a Ai\Big(\gamma(y - y_c)\Big) \tag{4.15}$$

is large in the critical layer (of order $O(\nu^{-1/4})$).

4.3 Green function for Orr Sommerfeld

Let us now construction the Green function $G(x,y)$ for Orr Sommerfeld equation (see [14] for a detailed study). We search G under the form

$$G = G_i + G_b$$

where G_i takes care of the source term δ_x and where G_b takes care of the boundary conditions. We look for $G_i(x,y)$ of the form

$$G_i(x,y) = a_+(x)\phi_{s,+}(y) + b_+(x)\frac{\phi_{f,+}(y)}{\phi_{f,+}(x)} \quad \text{for} \quad y < x,$$

$$G_i(x,y) = a_-(x)\phi_{s,-}(y) + b_-(x)\frac{\phi_{f,-}(y)}{\phi_{f,-}(x)} \quad \text{for} \quad y > x,$$

where $\phi_{f,\pm}(x)$ play the role of normalization constants. Let

$$F_\pm = \phi_{f,\pm}(x)$$

and let

$$v(x) = (-a_-(x), a_+(x), -b_-(x), b_+(x)).$$

By definition of a Green function, G_i, $\partial_y G_i$ and $\partial_y^2 G_i$ are continuous at $x = y$, whereas $-\varepsilon\partial_y^3 G_i$ has a unit jump at $x = y$. Let

$$M = \begin{pmatrix} \phi_{s,-} & \phi_{s,+} & \phi_{f,-}/F_- & \phi_{f,+}/F_+ \\ \partial_y\phi_{s,-} & \partial_y\phi_{s,+} & \partial_y\phi_{f,-}/F_- & \partial_y\phi_{f,+}/F_+ \\ \partial_y^2\phi_{s,-} & \partial_y^2\phi_{s,+} & \partial_y^2\phi_{f,-}/F_- & \partial_y^2\phi_{f,+}/F_+ \\ \partial_y^3\phi_{s,-} & \partial_y^3\phi_{s,+} & \partial_y^3\phi_{f,-}/F_- & \partial_y^3\phi_{f,+}/F_+ \end{pmatrix}, \tag{4.16}$$

where the various functions are evaluated at x. Then

$$Mv = (0,0,0,-\varepsilon^{-1}). \tag{4.17}$$

Let A, B, C and D be the two by two matrices defined by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

It can be proven (see [14]) that D is invertible, A is related to Rayleigh equation and that C is small. As a consequence,

$$\widetilde{M} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}$$

is an approximate inverse of M. By iteration, this allows to construct the true inverse of M and to get bounds on it [14].

We now add to G_i another Green function G_b to handle the boundary conditions. We look for G_b under the form

$$G_b(y) = d_s\phi_{s,-}(y) + d_f\frac{\phi_{f,-}(y)}{\phi_{f,-}(0)},$$

where $\phi_{f,-}(0)$ in the denominator is a normalization constant, and look for d_s and d_f such that

$$G_i(x,0) + G_b(0) = \partial_y G_i(x,0) + \partial_y G_b(0) = 0. \tag{4.18}$$

Let

$$N = \begin{pmatrix} \phi_{s,-} & \phi_{f,-}/\phi_{f,-}(0) \\ \partial_y \phi_{s,-} & \partial_y \phi_{f,-}/\phi_{f,-}(0) \end{pmatrix},$$

the functions being evaluated at $y = 0$. Then (4.18) can be rewritten as

$$Nd = -(G_i(x,0), \partial_y G_i(x,0))$$

where $d = (d_s, d_f)$. We have

$$N^{-1} = \frac{1}{\det(N)} \begin{pmatrix} \partial_y \phi_{f,-}(0)/\phi_{f,-}(0) & -1 \\ -\partial_y \phi_{s,-}(0) & \phi_{s,-}(0) \end{pmatrix},$$

which ends the construction.

4.4 Asymptotic behavior of linearised Navier Stokes

As for Euler equation, using contour integral, we have

$$e^{L_{NS} t} \psi_0 = \frac{\alpha}{2\pi} \int_\Gamma e^{i\alpha c t} Orr^{-1}_{\alpha,c,\nu} \psi_0 \, dc, \tag{4.19}$$

where

$$OS^{-1}_{\alpha,c,\nu}(y) = \int G(x,y) \psi_0(x) \, dx \tag{4.20}$$

where G is the Green function constructed in the previous paragraph. Combining these two identities we get

Theorem 4.1. *Let U_s be a monotonic and analytic shear layer profile, such that $U_s(0) = 0$ and such that $U_s(y)$ converges exponentially fast to some constant U_+ as $y \to +\infty$. Then for any small α,*

$$v_\alpha(t) = \exp(L_{NS,\alpha} t) P_{unstable,\alpha} v(0) + O(e^{-C\nu^{1/2} \langle t \rangle}), \tag{4.21}$$

where $P_{unstable,\alpha}$ is the projection on unstable modes.

Moreover, there exists C_0 and C_1 such that $P_{unstable,\alpha} = 0$ if $|\alpha| \le C_0 \nu^{1/4}$ or $|\alpha| \ge C_1 \nu^{1/6}$. For $C_0 \nu^{1/4} \le |\alpha| \le C_1 \nu^{1/6}$, then

$$P_{unstable,\alpha}(v_0) = e^{\lambda(\alpha) t} (v_0 | v(\alpha)) + c.c.,$$

where $v(\alpha)$ is an eigenvector of its adjoint $L^\star_{NS}$ with the same eigenvalue, and

$$\sup_\alpha |\Re \lambda(\alpha)| \sim C_2 \nu^{1/2}$$

as $\nu \to 0$.

The proof of this result is detailed in [3]. The main idea is that, thanks to the viscosity, there is no longer any singularity when c is on the range of U_s. We can thus shift Γ to negative $\Im c$. However, the Airy functions are increasing when the imaginary part of their argument is negative, and we can only shift Γ by $\nu^{1/4}$ in $\{\Im c < 0\}$, which gives a small decay rate, of magnitude $C\nu^{1/4}$. Note that, in strong contrast with linearised Euler equation, there exist growing modes for particular values of α: a shear monotonic shear layer is always linearly unstable provided the viscosity is small enough. This is somehow paradoxical: adding viscosity destabilises the flow.

5 Euler unstable shear layers

In this section, we begin the discussion of Theorem 1.1, and construct a sequence of solutions V^ν which exhibits the scale $\nu^{3/4}$. The construction of the scale $\nu^{13/16}$ will be done in section 7. The rigorous justification of this first step can be found in [16].

We note that V_0^ν has a vertical scale $\nu^{1/2}$. The first step is to rescale time and space to this scale, namely to introduce

$$T_1 = \frac{t}{\nu^{1/2}}, \qquad X_1 = \frac{x}{\nu^{1/2}}, \qquad Y_1 = \frac{y}{\nu^{1/2}}.$$

Navier Stokes remains unchanged by this change of variables, except that the viscosity is now

$$\nu_1 = \nu^{1/2}.$$

From now on we work in these rescaled variables. The starting point is a profile $U_s(0,\cdot)$ which is spectrally unstable for linearised Euler equations, namely a profile $U_s(0,\cdot)$ such that there exists α, $c_{1,0}$ and $\phi_{1,0}$, solutions of the Rayleigh equation

$$\Big(U_s(0,Y_1) - c_{1,0}\Big)(\partial_{Y_1}^2 - \alpha^2)\phi_{1,0} = U_s''(0,Y_1)\phi_{1,0}, \tag{5.1}$$

with

$$\phi_{1,0}(0) = 0 \tag{5.2}$$

and with $\Im c_{1,0} > 0$ (unstable mode).

We start from $(c_{1,0},\phi_{1,0})$ of (5.1) to construct a solution of Orr Sommerfeld equation and thus to build an instability for Navier Stokes equations. Note that $\phi_{1,0}$ does not fit the boundary condition $\partial_y\phi(0) = 0$. We therefore have to add a "boundary layer" to $\phi_{1,0}$ in order to ensure this boundary layer condition. Its size must be such that $\varepsilon\partial_{Y_1}^4$ balances the term in $\partial_{Y_1}^2$, which leads to a size of order $\varepsilon_1^{1/2}$ and to look for solutions of Orr Sommerfeld equations of the form

$$\phi(Y_1) = \sum_{n\geq 0} \varepsilon_1^{n/2}\phi_n^{int}(Y_1) + \sum_{n\geq 1}\varepsilon_1^{n/2}\phi_n^{bl}(\varepsilon_1^{-1/2}Y_1) \tag{5.3}$$

together with the expansion

$$c_1 = \sum_n \varepsilon_1^{n/2} c_{1,n}. \tag{5.4}$$

In this expansion ϕ_n^{int} refers to an "interior" behaviour, and ϕ_n^{bl} to a "boundary layer" behaviour. Moreover we start with $\phi_0^{int} = \phi_{1,0}$. The first boundary layer profile satisfies

$$\varepsilon_1\partial_{Y_1}^4\phi_1^{bl} = (U_s(0,0) - c_1)\partial_Y^2\phi_1^{bl}, \tag{5.5}$$

together with the boundary condition

$$\partial_{Y_1}\phi_1^{bl}(0) = -\partial_{Y_1}\phi_0^{int}(0). \tag{5.6}$$

Hence

$$\phi_1^{bl}(\varepsilon_1^{-1/2}Y_1) = -\partial_{Y_1}\phi_0^{int}(0)\varepsilon_1^{1/2}\mu^{-1}\Big(1 - e^{-\varepsilon_1^{-1/2}\mu Y_1}\Big), \qquad \mu = (-c_1)^{1/2}.$$

Let us now compute the corresponding velocity field. Using

$$(u^{bl}, v^{bl}) = \nabla_{X_1,Y_1}^\perp\Big(e^{i\alpha_1(X_1 - c_1T_1)}\phi^{bl}(\varepsilon_1^{-1/2}Y_1)\Big),$$

we see that u^{bl} and v^{bl} have the following asymptotic expansions

$$u^{bl} = -e^{i\alpha_1(X_1 - c_1 T_1)} e^{-\varepsilon_1^{-1/2}\mu Y_1} \partial_{Y_1}\phi_0^{int}(0) + \text{c.c.} + O(\varepsilon_1^{1/2})$$

and

$$v^{bl} = -i\alpha_1 \varepsilon_1^{1/2}\mu^{-1} e^{i\alpha_1(X_1 - c_1 T_1)} \Big(1 - e^{-\varepsilon_1^{-1/2}\mu Y_1}\Big) \partial_{Y_1}\phi_0^{int}(0) + \text{c.c.} + O(\varepsilon_1).$$

We note that the horizontal speed in the boundary layer is of order $O(1)$ and exactly compensates the interior component of the velocity at the boundary. The vertical speed is of size $O(\varepsilon_1^{1/2})$ and vanishes as $\nu \to 0$.

The construction of [15] exactly relies on this formal analysis and fully justifies it. It provides the existence of a solution V_1^ν of Navier Stokes equation with forcing term F^ν which is of the form $V_1^\nu = (u_1^\nu, u_2^\nu)$ with

$$u_1^\nu(t,x,y) = U_s(T, Y_1) + \nu^N e^{i\alpha_1(X_1 - c_1 T_1)} \partial_{Y_1}\phi_0^{int}(t, Y_1) - \nu^N e^{i\alpha_1(X_1 - c_1 T_1)} e^{-\varepsilon_1^{-1/2}\mu Y_1} \partial_{Y_1}\phi_0^{int}(0)$$
$$+\text{c.c.} + \cdots,$$
$$v_1^\nu = -i\nu^N \alpha_1 \varepsilon_1^{1/2}\mu^{-1} e^{i\alpha_1(X_1 - c_1 T_1)} \Big(1 - e^{-\varepsilon_1^{-1/2}\mu Y_1}\Big) \partial_{Y_1}\phi_0^{int}(0) + \text{c.c.} + \cdots.$$

This expansion may be justified till the perturbation reaches an order $O(1)$, namely become of the order of the shear flow itself.

Note that in the original y variable, the size of this new boundary layer is of order $\nu^{3/4}$. Therefore at this stage, we have *three* vertical scales:

- $O(1)$ (characteristic size of the domain). At this scale the flow is constant and equals $(U_+, 0)$.
- $O(\nu^{1/2})$ (characteristic size of the shear layer profile or boundary layer profile). At this scale the flow is the sum of $(U_s(Y_1), 0)$ and of an exponentially growing periodic perturbation, solution of Rayleigh equation. This instability creates a velocity at the boundary which periodically changes sign in space and time, in contradiction with Dirichlet boundary condition.
- $O(\nu^{3/4})$ (size of the sublayer of the unstable mode). The horizontal part of the velocity in this sublayer recovers Dirichlet boundary condition. The vertical one ensures incompressibility by creating a small flow of order $O(\nu^{1/4})$, which leaves or enters this sublayer.

There are *two* horizontal sizes, namely

- $O(1)$ (size of the domain),
- $O(\nu^{1/2})$ (horizontal periodicity of the instability). Note that this size is absent in Prandtl's boundary layer. Instability creates a small scale in x, which is absent from Prandtl's Ansatz. This scale is also "killed" if we assume analyticity of the initial data.

Moreover we have *two* times scales

- $O(1)$ (characteristic time scale of the evolution of U_s)
- $O(\nu^{1/2})$ (characteristic time scale of the instability in the boundary layer)

All these characteristic scales appear in classical physics books like [12]. As we will see in section 7, a third time scale will appear, linked with the instability of the sublayer of size $O(\nu^{3/4})$, together with a new and unexpected "sub-sub-layer".

6 Euler stable shear layers

In this section we discuss Theorem 1.2. As in the previous section we introduce T_1, X_1 and Y_1. However, as $U_s(0,\cdot)$ is stable for linear Euler equation, we can not build an instability starting from Rayleigh. Instead we use the instability built in subsection 4.2. For α_1 in the range

$$A_1 \nu_1^{1/4} \le \alpha_1 \le A_2 \nu_1^{1/6}$$

where A_1 and A_2 are two constants, there exists unstable modes of Orr Sommerfeld equations with a growing rate

$$\alpha_1 \Im c_1 \sim \nu_1^{1/2} \sim \nu^{1/4}.$$

This leads to instabilities within times $T_{insta,1}$ of order $\nu^{-1/4}$ in rescaled time, and therefore within real times of order

$$T_{insta,1} \sim \nu^{-1/4}\nu^{1/2} \sim \nu^{1/4} \ll 1.$$

Note that the unstable mode ψ_1 itself has a complex vertical structure, with a critical layer at y_c with $U_s^0(y_c) = c_1$, namely $y_c \sim \nu^{1/8}$ in rescaled variable, or $\nu^{1/8}\nu^{1/2} = \nu^{5/8}$ in original variables.. It also has a "recirculation" layer of size $\alpha_1^{-1} \sim \nu^{-1/4}$ in rescaled variable, or $\nu^{1/4}$ in original variables.

We note that there exist *four* vertical scales, namely

- $O(1)$, characteristic size of the domain,
- $O(\nu^{3/8})$, characteristic size of the recirculation layer,
- $O(\nu^{1/2})$, characteristic size of U_s (Prandtl's boundary layer size),
- $O(\nu^{5/8})$, characteristic size of the critical layer (new layer created by the instability),

There exist *two* horizontal scales

- $O(1)$, characteristic size of the domain
- $O(\nu^{1/8})$, periodicity of the instability arising in the layer.

There exits *two* time scales

- $O(1)$, characteristic time of the evolution of V^ν,
- $O(\nu^{1/8})$, characteristic time of the growth of the instability.

A crucial point is to investigate the size that this perturbations may reach. Perturbations as constructed in section 2, namely based on instabilities of Euler equations, reach a size $O(1)$. However the situation is more delicate for instabilities constructed in section 3, namely purely viscous ones, since their growth is very small, namely over times of order $\nu^{-1/8}$ in rescaled time.

Let $\phi(T)$ be the size of the instability. We observe that quadratic interactions have a different wavenumber, namely $\pm 2\alpha_1$, thus the feedback only occurs with cubic interactions, which have wave numbers $\pm\alpha_1$ and $\pm 3\alpha_1$. Thus, if we follow a bifurcation approach, $\phi(t)$ approximately satisfies an equation of the form

$$\dot\phi(T) = \alpha_1 c_1 \phi(T) + A|\phi(T)|^2\phi(T) + \cdots, \tag{6.1}$$

with $\Im(\alpha_1 c_1)$ or order $\nu^{1/8}$. The evolution of $\phi(T)$ then depends on the sign of $\Re A$.

- If $\Re A < 0$, then the cubic interactions tend to "tame" the instability. In this case it is possible to prove that $\phi(T)$ reaches a magnitude $\nu^{1/16}$, but not more.

- If $\Re A = 0$, then cubic interactions cancel, and we have to go up to quintic interactions. Then $\phi(T)$ is expected to react $\nu^{1/32}$.

- If $\Re A > 0$, then there is no obstacle for $\phi(T)$ to reach an order $O(1)$.

Note that A is the projection on the unstable mode of the cubic interaction of this unstable mode with itself. Preliminary numerical computations [2] indicate that we are in the first case. It is possible to prove that the perturbation reaches at least a size $O(\nu^{1/16})$. It therefore seems that we are in a bifurcation context, of Hopf's type, which may lead to the onset of rolls, a direction of research which deserves further investigations...

7 Multiple layers

We now turn to the discussion of the $\nu^{13/16}$ sublayer of Theorem 1.1 and go on with section 5. In the section 5 we have constructed a sublayer of size $\nu^{3/4}$ in which the typical velocity reaches a magnitude 1. We can construct a Reynolds number for the $\nu^{3/4}$ sublayer by combining a typical length (its size, namely $\nu^{3/4}$), a typical velocity (the magnitude of V_1^ν, namely U_+) and the viscosity (ν). This leads to a Reynolds number for this sublayer of order

$$Re_2 \sim \frac{\nu^{3/4} U_+}{\nu} \sim \nu^{-1/4}.$$

Hence, the sublayer sees its Reynolds number go to infinity as ν goes to 0. But any shear layer is linearly unstable at high Reynolds number. Thus, we expect linear instabilities to appear in this sublayer.

To study these instability we consider V^ν, as constructed in section 5, at a time T_1^ν where the $\nu^{3/4}$ layer has fully developed, i.e. has reached a magnitude $O(1)$. We consider this time as the new initial time. We then rescale time and space in order to focus on the $\nu^{3/4}$ layer, and introduce

$$T_2 = \nu^{-3/4} t, \quad X_2 = \nu^{-3/4} x, \quad Y_2 = \nu^{-3/4} y.$$

After rescaling, Navier Stokes equations remain unchanged, excepted for the viscosity which is now

$$\nu_2 = \nu^{1/4} \sim \frac{1}{Re_2}.$$

Let us now describe the flow $V^\nu(T_1^\nu, \cdot, \cdot)$ near $x = 0$, in the rescaled variables X_2 and Y_2, namely let us describe $V^\nu(T_1^\nu, \nu^{3/4} X_2, \nu^{3/4} Y_2)$. We recall that $V^\nu(T_1^\nu, \cdot, \cdot)$ has scales 1 and $\nu^{1/4}$ in x and $\nu^{1/2}$ and $\nu^{3/4}$ in y. Hence, in the variables X_2 and Y_2, $V^\nu(T_1^\nu, \cdot, \cdot)$ changes on scales of order $\nu^{-1/4}$ in the X_2 variable, namely is almost constant. In the Y_2 variable it has the scales 1 and $\nu^{-1/4}$. In other words

$$V^\nu(T_1^\nu + \nu^{3/4} T_2, \nu^{3/4} X_2, \nu^{3/4} Y_2) = U_1 + O(\nu^{1/4} X_2 + \nu^{1/4} Y_2) \tag{7.1}$$

where

$$U_1 = (U_1^\nu, 0),$$

with

$$U_1^\nu(X_2, Y_2) = -e^{i\alpha_1 \nu^{1/4}(X_2 - c_2 T_2)} \Big[1 - e^{-\mu Y_2} \Big] \partial_{Z_1} \phi_0^{int}(0) + \text{c.c.}.$$

Note that the boundary layer has an exponential behaviour in Y_2, and periodically depends on the X_2 variable, with a very large spatial period of order $O(\nu^{-1/4})$ in this variable. We note that U_1^ν

has no inflection point and is therefore stable for Rayleigh's equation. Moreover, U_1^ν depends on X_2, which was not the case in section 5 and 6.

The first step is to "freeze" the X_2 variable at $X_2 = 0$ and to look for an unstable mode of the form

$$v = \nabla^\perp \Big(e^{\alpha_2 (X_2 - c_2 T_2)} \psi_2(Y_2) \Big)$$

for the profile $U_1^\nu(0, Y_2)$. Then ψ_2 must satisfy the following Orr Sommerfeld equations

$$\varepsilon_2 (\partial_{Y_2}^2 - \alpha_2^2)^2 \psi_2 = (U_1^\nu(0, Y_2) - c_2)(\partial_{Y_2}^2 - \alpha_2^2)\psi_2 - \partial_{Y_1}^2 U_1^\nu(0, Y_2)\psi_2. \tag{7.2}$$

As proved in [16] and recalled in section 6, for α_2 in the range

$$A_1 \nu_2^{1/4} \le \alpha_2 \le A_2 \nu_2^{1/6}$$

where A_1 and A_2 are two constants, there exists unstable modes to (7.2) with a growth rate

$$\alpha_2 \Im c_2 \sim \nu_2^{1/2} \sim \nu^{1/8}.$$

This leads to instabilities within times $T_{insta,2}$ of order $\nu^{-1/8}$ in rescaled time, and therefore within real times of order

$$T_{insta,2} \sim \nu^{-1/8} \nu^{3/4} \sim \nu^{5/8} \ll \nu^{1/2}.$$

Hence the instabilities of this sublayer grow faster than the initial Prandtl instability. Note that $T_{insta,2}$ is large in T_2 scale. In the rescaled variable T_2, the growth of the instability is slow, mainly because this instability has large structures in X_2. Let us choose α_2 of order $\nu_2^{1/4} \sim \nu^{1/16}$ to fix the ideas. This corresponds to structures in the horizontal variable of size $\nu^{-1/16}$ in X_2 variable, or $\nu^{3/4 - 1/16} = \nu^{11/16}$ in the initial x variable. Note that c_2 is of order $\nu_2^{1/4} \sim \nu^{1/16}$.

Note that this instable mode ψ_2 itself has a complex vertical structure, composed of three scales. Of course $Y_2 \sim 1$, but also a "critical layer" and a "recirculation layer". First the "critical layer" is defined by

$$U_1^\nu(Z_c) = c_2.$$

We note that

$$Z_c \sim \nu^{1/16},$$

or, in initial variables,

$$z_c \sim \nu^{3/4} \nu^{1/16} \sim \nu^{13/16}.$$

There is also a "recirculation layer", of size α_2^{-1} in Y_2 variable, corresponding to the kernel of $\partial_{Y_2}^2 - \alpha_2^2$. It size is $\nu_2^{-1/4} \nu^{3/4} = \nu^{-1/16} \nu^{3/4} = \nu^{11/16} \ll \nu^{1/2}$. It is thus smaller that Prandtl's layer. Note that this vertical scale corresponds to the scale in X_2: the instability creates large scale rolls to handle the flow in the critical layer.

In the previous paragraph we have constructed an instability for $(U_1^\nu, 0)$ and not the real layer (U_1^ν, V_1^ν). We have now to check whether the instability can survive to the spatial dependency of the background, and to the small upward velocity. The difference between U_1 and (U_1^ν, V_1^ν) has two aspects: we have a dependence on $\nu^{1/4} Y_2$ and an additional vertical velocity V_1^ν is of order $\nu^{1/4}$.

Let us discuss the first aspect. We will pick up the most unstable mode α_2, namely the mode with the largest growth rate $\alpha_2 \Im c_2$, and construct a "wave packet" of instabilities near this α_2. Our analysis of growth is valid as long as this wave packet does not travel to areas where the underlying flow notably changes. Hence it is only valid as long as it travels on length scales much smaller than $\nu^{-1/4}$ in rescaled variables, hence on length scales much smaller than $\nu^{1/2}$ in original variables.

The wave packet travels with group velocity

$$\sigma_2 = \frac{\partial(\alpha_2 c_2)}{\partial \alpha_2} = c_2 + \alpha_2 \frac{\partial c_2}{\partial \alpha_2} \sim \nu_2^{1/4} \sim \nu^{1/16},$$

according to Appendix A. The maximum travel distance, within the instability time $T_{insta,2}$, is

$$Y_2 = \sigma_2 T_{insta,2} \approx \nu^{1/16} \nu^{-1/8} = \nu^{-1/16},$$

or in the original coordinates, the maximum travel distance is

$$\nu^{3/4} \nu^{-1/16} = \nu^{11/16} \ll \nu^{1/2}.$$

As a consequence we may "localise" the instability and construct an instability for the real layer (U_1^ν, V_1^ν) starting from an instability of $(U_1^\nu, 0)$ (see [5] for a rigorous proof). This ends the discussion of Theorem 1.1. It turns out that this instability is too small in magnitude to allow the construction of an other sublayer, and to iterate the construction.

8 Discussion

As mentioned in the introduction, the various results on the behavior of solutions to Navier Stokes equations at low viscosity are somehow paradoxical.

A first series of results states that Prandtl's analysis is true in small time and for analytic data. This is exactly the physical statement. However the smoothness of the data is too strong, and hides the underlying existing instability. Let us give a simple example. Let us consider the complex transport equation

$$\partial_t \phi + i \partial_x \phi = 0.$$

Then the Fourier transform satisfies

$$\partial_t \phi(t, \xi) = \xi \phi(t, \xi), \tag{8.1}$$

and of course

$$\phi(t, \xi) = e^{\xi t} \phi(0, \xi).$$

In particular (8.1) is ill posed in any Sobolev space. However if we introduce the analytic norm

$$\|\phi(t, \cdot)\|_\sigma = \sup |\phi(t, \xi)| \; e^{-\sigma |\xi|}$$

then we see that

$$\|\phi(t, \cdot)\|_{\sigma - t} \le \|\phi(0, \cdot)\|_\sigma,$$

hence ϕ is locally well posed in analytic space. The analyticity simply "kills" the instability, which is hidden until $t = \sigma$ where the radius of analyticity vanishes.

Let us now discuss the multilayers instability. A first remark is that a shear layer is unstable provided its Reynolds number is large enough, usually of order $1000 - 10000$ for an exponential profile. We note that the $\nu^{3/4}$ appears if $Re_1 > 1000 - 10000$, which means that the initial Reynolds number is larger than $10^6 - 10^8$. The third layer appears if $Re_2 > 1000 - 10000$, namely if the Reynolds number is of order $10^9 - 10^{12}$, namely at very high Reynolds numbers. In practise it is impossible to observe laminar flows at such Reynolds number, since small perturbations on the walls would immediately lead to a turbulent behaviour.

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