Concentrated vortex rings for Euler and Navier-Stokes equations

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We consider the evolution of the vorticity initially concentrated on a torus of small section (i.e. *vortex ring*). Let us analyse first the Euler flow

$$\partial_t \boldsymbol{\omega} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} \,,$$

$$\boldsymbol{u}(\boldsymbol{\xi}, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{\eta} \, \frac{(\boldsymbol{\xi} - \boldsymbol{\eta}) \wedge \boldsymbol{\omega}(\boldsymbol{\eta}, t)}{|\boldsymbol{\xi} - \boldsymbol{\eta}|^3} \,.$$

We take initial data such that the velocity field is *axisymmetric without swirl*, that is, introducing cylindrical coordinates (z, r, θ),

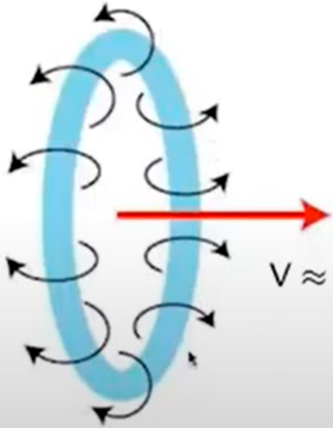
$$\boldsymbol{e}_z = (1, 0, 0) \quad \text{“horizontal”}$$

$$\boldsymbol{e}_r = (0, \cos\theta, \sin\theta) \quad \text{“radial”}$$

$$\boldsymbol{e}_\theta = (0, -\sin\theta, \cos\theta) \quad \text{“toroidal”}$$

$$\boldsymbol{u} = u_z(z, r, t)\boldsymbol{e}_z + u_r(z, r, t)\boldsymbol{e}_r$$

$$\boldsymbol{\omega} = \omega_\theta(z, r, t)\boldsymbol{e}_\theta \,, \quad \omega_\theta = \partial_z u_r - \partial_r u_z$$



$$V \approx \frac{\Gamma}{4\pi R} \log \frac{R}{r}$$

The incompressibility condition writes

$$\operatorname{div} \boldsymbol{u} = \partial_z u_z + \partial_r u_r + \frac{1}{r} u_r = 0 \,.$$

In terms of the component $\omega_\theta := \omega$ the evolution equation for ω reads

$$\partial_t \omega + (u_z \partial_z + u_r \partial_r)\omega - \frac{u_r \omega}{r} = 0$$

and

$$u_z = -\frac{1}{2\pi} \int \mathrm{d}z' \int_0^\infty r' \mathrm{d}r' \int_0^\pi \mathrm{d}\theta \, \frac{\omega(z', r', t)(r\cos\theta - r')}{[(z-z')^2 + (r-r')^2 + 2rr'(1-\cos\theta)]^{3/2}}$$

$$u_r = \frac{1}{2\pi} \int \mathrm{d}z' \int_0^\infty r' \mathrm{d}r' \int_0^\pi \mathrm{d}\theta \, \frac{\omega(z', r', t)(z-z')\cos\theta}{[(z-z')^2 + (r-r')^2 + 2rr'(1-\cos\theta)]^{3/2}}$$

this means that ω is conserved along the flow,

$$\frac{\omega(z(t), r(t), t)}{r(t)} = \frac{\omega(z(0), r(0), 0)}{r(0)}$$

with $(z(t), r(t))$ solution to

$$\dot{z}(t) = u_z(z(t), r(t), t)\,, \qquad \dot{r}(t) = u_r(z(t), r(t), t)$$

The initial value problem for the axisymmetric solution to the Euler (and Navier-Stokes) equation has been studied since the pioneering papers of

Ladyzhenskaya, *Zapisky Nauchnych Sem.* (1968)

Ukhovskii and Yudovitch, *J. Appl. Math. Mech.* (1968) and more recently by

Feng and Šverák, *Arch. Ration. Mech. Anal.* (2015)

Gallay and Šverák, *Confluentes Math.* (2015), *Ann. Sci. Éc. Norm. Supér.* (2019), preprint (2023)

regarding (for N-S) an initial vorticity either integrable or a circular vortex filament.

Here we focus on the following property of the solution: one (or more) vortex ring evolves with uniform velocity parallel to the symmetry axis in the limit of high concentration.
Such behavior reflects what happens also in the stationary case, as proved first rigorously by

Fraenkel, *Proc. Roy. Soc. Lond. A.* (1970)

Fraenkel and Berger, *Acta Math.* (1974)

More precisely, if the vorticity is supported in an annulus of thickness ε and fixed radius (distance from the symmetry axis), and the vorticity mass vanishes as $|\log \varepsilon|^{-1}$, then in the limit $\varepsilon \to 0$ the vorticity remains concentrated in a thin annulus which performs a rectilinear motion along the symmetry axis.

Such scaling of the vorticity mass as $|\log \varepsilon|^{-1}$ is due to have a non-trivial velocity, i.e. not vanishing either diverging

$$V \approx \frac{1}{4\pi r_0} \log \frac{r_0}{\varepsilon} \int \mathrm{d}z \mathrm{d}r \, \omega(z, r, 0)$$

In case the distance of the ring from the symmetry axis is larger, i.e. it diverges when $\varepsilon \to 0$, we have a different behavior.

When $r_0 \approx |\log \varepsilon|$ and the vorticity mass is $O(1)$, then for one ring alone it performs a uniform rectilinear motion, see
Marchioro and Negrini, *NoDEA* (1999)
whereas for many rings (interacting each other) we conjecture that the motions of the centers of vorticity converge to a dynamical system which is

$$\dot{\boldsymbol{x}}_i(t) = -\frac{1}{2\pi} \sum_{j=1, j\neq i}^{N} a_j \nabla_i^{\perp} \log |\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)| + a_i \boldsymbol{e}_1$$

with $\boldsymbol{x}_i \in \mathbb{R}^2$, $\boldsymbol{e}_1 = (1, 0)$. It is a "modified" point vortex system, which describes the so called *leapfrogging* phenomenon.

When $r_0 \approx |\log \varepsilon|^\alpha$, for $\alpha > 2$ (or a faster growth for $\varepsilon \to 0$) the convergence is to the classical point vortex system (as in the planar case), see
Cavallaro and Marchioro, *J. Math. Phys.* (2021).

Very recently
Dávila, Del Pino, Musso, and Wei, *arXiv:2207.03263v3* (2023)
derive rigorously the leapfrogging dynamics from a different scaling: they consider N coaxial vortex rings at finite distance from the symmetry axis, with a mutual distance of order $|\log \varepsilon|^{-1/2}$.

We sketch now the results of the paper
Buttà, Cavallaro, and Marchioro, *ZAMP* (2022).
An equivalent weak formulation of the Euler equation in terms of vorticity is obtained from the previous one by an integration by parts

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t[f] = \omega_t[u_z\partial_z f + u_r\partial_r f + \partial_t f]$$

where $f = f(z, r, t)$ is any bounded smooth test function and

$$\omega_t[f] := \int \mathrm{d}z \int_0^\infty \mathrm{d}r\, \omega(z, r, t) f(z, r, t)$$

considering an initial bounded vorticity $\omega(z, r, 0)$ with compact support contained in the open half-plane $\Pi := \{(z, r) : r > 0\}$.
A point in Π corresponds to a circumference in $\mathbb{R}^3$.

We fix the initial data: given $\varepsilon \in (0, 1)$

$$\omega_\varepsilon(z, r, 0) = \sum_{i=1}^{N} \omega_{i,\varepsilon}(z, r, 0)$$

where $\omega_{i,\varepsilon}(z, r, 0)$, $i = 1, \dots, N$, are functions with definite sign whose support

$$\Lambda_{i,\varepsilon}(0) := \operatorname{supp} \omega_{i,\varepsilon}(\cdot, 0) \subset \Sigma(\zeta^i|\varepsilon)$$

with

$$\overline{\Sigma(\zeta^i|\varepsilon)} \subset \Pi \quad \forall\, i\,, \qquad \Sigma(\zeta^i|\varepsilon) \cap \Sigma(\zeta^j|\varepsilon) = \emptyset \quad \forall\, i \neq j$$

for fixed $\zeta^i = (z_i, r_i) \in \Pi$. We assume also that

$$\min_i r_i > 2D \quad \forall\, i\,, \qquad |r_i - r_j| \geq 2D \quad \forall\, i \neq j$$

where D is a positive fixed constant. This means that the annuli have different radii, which is an essential hypothesis.

Such decomposition extends to positive time setting

$$\omega_\varepsilon(z,r,t) = \sum_{i=1}^{N} \omega_{i,\varepsilon}(z,r,t)$$

with $\omega_{i,\varepsilon}(x,t)$ the time evolution of the ith vortex ring,

$$\omega_{i,\varepsilon}(z(t),r(t),t) := \frac{r(t)}{r(0)}\omega_{\varepsilon,i}(z(0),r(0),0)\,.$$

To have a finite speed for the ith vortex ring we take the following normalization for the vorticity mass: we assume that there are N real parameters $a_1,\dots,a_N$, called *vortex intensities*, such that

$$|\log\varepsilon|\int \mathrm{d}z \int_0^\infty \mathrm{d}r\,\omega_{i,\varepsilon}(z,r,0) = a_i \quad \forall\, i=1,\dots,N\,.$$

Finally, to avoid too large vorticity concentrations, we further assume there is a constant $M>0$ such that

$$|\omega_{i,\varepsilon}(z,r,0)| \le \frac{M}{\varepsilon^2|\log\varepsilon|} \quad \forall\,(z,r)\in\Pi \quad \forall\, i=1,\dots,N\,.$$

Theorem

Assume the previous initial data $\omega_\varepsilon(x,0)$, and define

$$\zeta^i(t) := \zeta^i + \frac{a_i}{4\pi r_i} \begin{pmatrix} 1 \\ 0 \end{pmatrix} t\,, \quad i = 1, \dots, N\,.$$

Then, for any $T > 0$ the following holds true. For any ε small enough there are $\zeta^{i,\varepsilon}(t) \in \Pi$, $t \in [0, T]$, $i = 1, \dots, N$, and $R_\varepsilon > 0$ such that $\forall\, t \in [0, T]$

$$\lim_{\varepsilon \to 0} |\log \varepsilon| \int_{\Sigma(\zeta^{i,\varepsilon}(t)|R_\varepsilon)} \mathrm{d}z\, \mathrm{d}r\, \omega_{i,\varepsilon}(z, r, t) = a_i \quad \forall\, i = 1, \dots, N$$

with

$$\lim_{\varepsilon \to 0} R_\varepsilon = 0, \qquad \lim_{\varepsilon \to 0} \zeta^{i,\varepsilon}(t) = \zeta^i(t) \quad \forall\, t \in [0, T]\,.$$

Some remarks:

The assumption $|r_i - r_j| \geq 2D$ allows to prove that the supports $\Lambda_{i,\varepsilon}(t) \cap \Lambda_{i,\varepsilon}(t) = \emptyset$ for any $i \neq j$ and $t \geq 0$, by a strong localization property on each support that we are able to prove along the r-direction (the same it is not true along the z-direction). The prevention of an overlap of the supports it is crucial for our technique.

Our analysis concerns an asymptotic regime $\varepsilon \to 0$ in which the interaction between different rings actually vanishes, so that the limiting motion of each ring is not influenced by the other ones. Features of the dynamics for finite ε, as the leapfrogging phenomenon, although very interesting, are not described by our method.

The necessity to fix an (arbitrary) maximum time T comes from the application of a concentration result stated in
Benedetto, Caglioti, and Marchioro, *Math. Meth. Appl. Sci* (2000)
in which some uniform bounds on the axial motion require a finite translation in this direction.

It should be investigated the possibility to let $T \approx |\log \varepsilon|$ as done (in the planar case) in
Buttà and Marchioro, *SIAM J. Math. Anal.* (2018)
in which the planar symmetry allows to get a good estimate on the moment of inertia.

Strategy of the proof:

We first show the corresponding result for a "reduced system", where a vortex ring alone moves under the action of a suitable external time-dependent vector field. The result for the original model is then achieved by treating the motion of each vortex ring as that of a reduced system, in which the external field describes the force due to its interaction with the other rings.

The key tool in the planar case is a sharp a priori estimate on the moment of inertia, which is not available in the axial symmetric case because the velocity field is not a Lipschitz function.

A suitable decomposition of the velocity field shows that its non Lipschitz part is directed along the z-axis, which suggests that the vorticity should stay more localized along the radial direction, Indeed, this is true and allows us to deduce an estimate on a different quantity, the "axial moment of inertia", which makes possible to build up an iterative scheme convergent at any positive time, thus deducing a sharp localization property along the radial direction globally in time.

Reduction to a single vortex problem.

We put $x = (x_1, x_2) := (z, r)$. The equations of motion take the following form,

$$u(x,t) = \int \mathrm{d}y\, H(x,y)\,\omega_\varepsilon(y,t)\,,$$

$$\omega_\varepsilon(x(t),t) = \frac{x_2(t)}{x_2(0)}\omega_\varepsilon(x(0),0)\,,$$

$$\dot{x}(t) = u(x(t),t) + F^\varepsilon(x(t),t)\,,$$

where $u(x,t) = (u_1(x,t), u_2(x,t))$ and the kernel $H(x,y) = (H_1(x,y), H_2(x,y))$ is given by

$$H_1(x,y) = \frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta\, \frac{y_2(y_2 - x_2\cos\theta)}{\left[|x-y|^2 + 2x_2y_2(1-\cos\theta)\right]^{3/2}}\,,$$

$$H_2(x,y) = \frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta\, \frac{y_2(x_1 - y_1)\cos\theta}{\left[|x-y|^2 + 2x_2y_2(1-\cos\theta)\right]^{3/2}}\,.$$

The external field $F^{\varepsilon}(x(t), t)$ is introduced to simulate the action of the other vortices on a given one.

Hence it is taken continuous, Lipschitz, and divergence free.

The first two assumptions derive from the fact that the supports of the $\omega_{i,\varepsilon}$ (in the original model) are disjointed, hence the velocity field produced by $\omega_{i,\varepsilon}$ in the region where $\omega_{j,\varepsilon}$ is supported (for $i \neq j$) is regular.

The assumptions on $\omega_{\varepsilon}(x, 0)$ are the same as for $\omega_{i,\varepsilon}$ given before.

Theorem

Let

$$\zeta(t) = \zeta^0 + \frac{a}{4\pi r_0}\begin{pmatrix}1\\0\end{pmatrix} t\,.$$

Then, for each $T > 0$ *the following holds true.*

- *For any* $k \in \left(0, \frac{1}{4}\right)$ *there is* $C_k > 0$ *such that, for any* ε *small enough and* $\forall\, t \in [0, T]$

$$\Lambda_\varepsilon(t) := \operatorname{supp}\omega_\varepsilon(x,t) \subset \{x \in \mathbb{R}^2 \colon |x_2 - r_0| \le C_k |\log\varepsilon|^{-k}\}\,.$$

- *For any* ε *small enough there are* $\zeta^\varepsilon(t) \in \Pi$ *and* $\varrho_\varepsilon > 0$ *such that* $\forall\, t \in [0, T]$

$$\lim_{\varepsilon\to 0} |\log\varepsilon| \int_{\Sigma(\zeta^\varepsilon(t)|\varrho_\varepsilon)} \mathrm{d}x\, \omega_\varepsilon(x,t) = a\,,$$

with

$$\lim_{\varepsilon\to 0} \varrho_\varepsilon = 0, \quad \lim_{\varepsilon\to 0} \zeta^\varepsilon(t) = \zeta(t)\,.$$

Remark: *notice that the Theorem describing the original model of N vortex rings follows easily by the one of the Reduced System.*

We give a sketch of the proof of the last Theorem.

We analyse separately the radial and the axial motion.
The kernel $H(x, y)$ can be split as

$$H(x,y) = K(x-y) + L(x,y) + \mathcal{R}(x,y)\,,$$

$$K(x) = \nabla^\perp G(x)\,, \quad G(x) := -\frac{1}{2\pi}\log|x|\,,$$

where $v^\perp := (v_2, -v_1)$ for $v = (v_1, v_2)$,

$$L(x,y) = \frac{1}{4\pi x_2}\log\frac{1+|x-y|}{|x-y|}\begin{pmatrix}1\\0\end{pmatrix}$$

and $\mathcal{R}(x, y)$ is bounded by

$$|\mathcal{R}(x,y)| \le C_0\frac{1 + x_2 + \sqrt{x_2 y_2}\,(1 + |\log(x_2 y_2)|)}{x_2^2}\,.$$

Thus the velocity field can be decomposed accordingly

$$u(x,t) = \widetilde{u}(x,t) + \int \mathrm{d}y\, L(x,y)\, \omega_\varepsilon(y,t) + \int \mathrm{d}y\, \mathcal{R}(x,y)\, \omega_\varepsilon(y,t)\,,$$

where $\widetilde{u}(x,t) = \int \mathrm{d}y\, K(x-y)\, \omega_\varepsilon(y,t)$.

Lemma

The following estimates hold true,

$$\int \mathrm{d}y\, |L(x,y)|\, \omega_\varepsilon(y,t) \le C\,, \quad \int \mathrm{d}y\, |\mathcal{R}(x,y)|\, \omega_\varepsilon(y,t) \le \frac{C}{|\log \varepsilon|}\,.$$

We denote by $B_\varepsilon(t) = (B_{\varepsilon,1}(t), B_{\varepsilon,2}(t))$ the center of vorticity of the blob, defined by

$$B_\varepsilon(t) = \frac{\int \mathrm{d}x\, x\, \omega_\varepsilon(x,t)}{\int \mathrm{d}x\, \omega_\varepsilon(x,t)} = |\log \varepsilon| \int \mathrm{d}x\, x\, \omega_\varepsilon(x,t)\,,$$

and by $I_\varepsilon(t)$ the *axial moment of inertia* with respect to $x_2 = B_{\varepsilon,2}(t)$, i.e.,

$$I_\varepsilon(t) = \int \mathrm{d}x\, \left(x_2 - B_{\varepsilon,2}(t)\right)^2 \omega_\varepsilon(x,t)\,.$$

We compute their time derivative, and obtain

$$|\dot{B}_{\varepsilon,2}(t)| \leq \frac{C}{|\log \varepsilon|},$$

$$|\dot{I}_{\varepsilon}(t)| \leq \frac{C}{|\log \varepsilon|^{3/2}}\sqrt{I_{\varepsilon}(t)} + \frac{C}{|\log \varepsilon|^{2}}.$$

By integration of the last differential inequality, since the initial data imply $I_{\varepsilon}(0) \leq 4\varepsilon^2$, we get

$$I_{\varepsilon}(t) \leq \frac{C}{|\log \varepsilon|^{2}}.$$

This bound will be an essential tool in the following.

In
Buttà and Marchioro, *J. Math. Fluid Mech.* (2020)
a more sophisticated argument allows to obtain the same estimate $C/|\log \varepsilon|^2$ for the moment of inertia with respect to the center of vorticity, but the whole argument works only for short times.

In our case, we are able to extend the analysis to any positive time, thanks to the previous estimate on $I_\varepsilon(t)$ and to the absence of the non Lipschitz term $L(x, y)$ when we analyze separately the motion along the radial direction.

The next Lemmas are preliminary to establish the property of compact support along the radial direction.

Lemma

$$R_t := \max\{|x_2 - B_{\varepsilon,2}(t)| \colon x \in \Lambda_\varepsilon(t)\}\,.$$

Given $x_0 \in \Lambda_\varepsilon(0)$, *let* $x(x_0,t)$ *be the fluid particle with initial condition* $x(x_0,0) = x_0$ *and suppose at time* t *it happens that*

$$|x_2(x_0,t) - B_{\varepsilon,2}(t)| = R_t\,.$$

Then, at this time t,

$$\frac{\mathrm{d}}{\mathrm{d}t}|x_2(x_0,t) - B_{\varepsilon,2}(t)| \le \frac{C}{|\log\varepsilon|} + \frac{1}{\pi R_t|\log\varepsilon|} + \sqrt{\frac{C m_t(R_t/2)}{\varepsilon^2|\log\varepsilon|}}\,,$$

where the function $m_t(h)$ *is defined by*

$$m_t(h) = \int_{|y_2 - B_{\varepsilon,2}(t)|>h} \mathrm{d}y\,\omega_\varepsilon(y,t)\,.$$

Lemma

For each $\ell > 0$ *and* $k \in \left(0, \frac{1}{4}\right)$,

$$\lim_{\varepsilon \to 0} \max_{t \in [0,T]} \varepsilon^{-\ell} m_t \left(\frac{1}{|\log \varepsilon|^k} \right) = 0\,.$$

This Lemma, proved by an iterative method, states that the vorticity mass contained in the region

$$|y_2 - B_{\varepsilon,2}(t)| > \frac{1}{|\log \varepsilon|^k}$$

(i.e. outside the strip centered in $B_{\varepsilon,2}(t)$ and of width $2|\log \varepsilon|^{-k}$) is indeed smaller than *any* power of ε.
A further argument shows that it is effectively zero.

Where does the upper bound $k < \frac{1}{4}$ come from?

In the proof of the previous Lemma one obtains an iterative scheme which, after $n = \mathrm{Intg}[|\log \varepsilon|]$ steps, gives

$$m_t(R_0) \le \frac{\left(C|\log \varepsilon|^{4k} t\right)^n}{n!}$$

with

$$R_0 = \frac{1}{|\log \varepsilon|^k} .$$

Hence, by Stirling formula, $m_t(R_0) \le \varepsilon^\ell$ for any $\ell > 0$ and for any positive time only if $k < \frac{1}{4}$.

In

Buttà and Marchioro, *J. Math. Fluid Mech.* (2020)

they obtain convergence only for short times, since it is considered the spread of the support of ω_ε both in r and z directions. Thus the presence of the term $L(x, y)$ forces to stop within short times.

Properties of the axial motion.

We extend to our context a Lemma which is proved in
Benedetto, Caglioti, and Marchioro, *Math. Meth. Appl. Sci* (2000)

Lemma

With the same initial conditions stated above, for each $T > 0$*, there are* $\varepsilon_1 \in (0,1)$*,* $C_1 > 0$ *and* $q_\varepsilon(t) \in \mathbb{R}^2$*, such that*

$$|\log \varepsilon| \int_{\Sigma(q_\varepsilon(t), \varepsilon |\log \varepsilon|)} \mathrm{d}x \, \omega_\varepsilon(x,t) \geq 1 - \frac{C_1}{\log|\log \varepsilon|}$$

$\forall\, t \in [0, T]$ *and* $\forall\, \varepsilon \in (0, \varepsilon_1]$*. Moreover*

$$q_\varepsilon(t) \xrightarrow{\varepsilon \to 0} \begin{pmatrix} z_0 \\ r_0 \end{pmatrix} + \frac{a}{4\pi r_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \, .$$

This concentration result, proved in [BCM2000] for one vortex ring alone, is extended to our Reduced System (one vortex ring in an external field).

The key ingredients to prove the previous concentration result, for a vortex ring alone, are the following constants of motion

$$M_0 = \int \mathrm{d}x \, \omega_\varepsilon(x, t)$$

$$M_2 = \int \mathrm{d}x \, x_2^2 \, \omega_\varepsilon(x, t)$$

$$E = \frac{1}{2} \int \mathrm{d}x \, 2\pi x_2 |u(x, t)|^2 \, .$$

In presence of an external field $F^\varepsilon(x, t)$ the quantity M_0 is still conserved, while M_2 and E are not.

By a direct computation, the time derivatives of M_2 and E are much smaller than their initial value, so they remain "almost" constant and the concentration result is valid.

We use this result to infer the properties of the motion of the vortex ring along the axial direction.
In particular, by some refined estimates, we conclude that

$$B_\varepsilon(t) \xrightarrow{\varepsilon\to 0} \begin{pmatrix} z_0 \\ r_0 \end{pmatrix} + \frac{a}{4\pi r_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} t\,.$$

This is achieved also thanks to the (complete) moment of inertia with respect to the center of vorticity

$$J_\varepsilon(t) = \int \mathrm{d}x\, |x - B_\varepsilon(t)|^2 \omega_\varepsilon(x,t)\,.$$

which can be bounded as

$$J_\varepsilon(t) \le \frac{C}{|\log\varepsilon|} \quad \forall\, t\in[0,T]\,,$$

This bound is weaker than the one for the axial moment of inertia ($C/|\log\varepsilon|^2$) and not enough to make work a previous iterative method. But it is sufficient to conclude the convergence of $B_{\varepsilon,1}(t)$ to a uniform motion.

Case with small viscosity

We want to add now a small viscosity ν, and study the joint limit for which $\varepsilon \to 0$, $\nu \to 0$, in such a way that $\nu \leq \varepsilon^2 |\log \varepsilon|^\gamma$ with $\gamma \in (0, 1)$.

In

Gallay and Šverák, preprint (2023)

the authors are able to study the vanishing viscosity limit for one initial vortex filament (which "corresponds" to $\varepsilon = 0$, $\nu \to 0$).

We consider in

Buttà, Cavallaro, and Marchioro, *J. Math. Phys.* (2022)

the same framework of the inviscid case. The main difficulty to tackle is the fact that the supports of the ω_i overlap at $t > 0$, and this requires much care to define a Reduced System (one vortex in an external field).

The external field (which simulates the other vortices) can be singular.

We underline the main differences with the inviscid case. The equation for ω is

$$\partial_t\omega + (u_z\partial_z + u_r\partial_r)\omega - \frac{u_r\omega}{r} = \nu\left[\partial_z^2\omega + \frac{1}{r}\partial_r\left(r\partial_r\omega\right) - \frac{\omega}{r^2}\right]$$

whereas in terms of the quantity ω/r the previous equation reads

$$\left[\partial_t + u_z\partial_z + \left(u_r - \frac{3\nu}{r}\right)\partial_r\right]\left(\frac{\omega}{r}\right) = \nu\left(\partial_z^2 + \partial_r^2\right)\left(\frac{\omega}{r}\right),$$

which means that ω/r evolves as a diffusion with drag, and it admits a "maximum principle", i.e. ω/r reaches its maximum initially.

By an integration by parts, the following weak formulation of the previous equation holds true,

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t[f] = \omega_t\left[u_z\partial_z f + u_r\partial_r f + \partial_t f\right] + \nu\omega_t\left[\partial_z^2 f + \partial_r^2 f - \frac{1}{r}\partial_r f\right],$$

where

$$\omega_t[f] := \int \mathrm{d}z \int_0^\infty \mathrm{d}r\, \omega(z,r,t) f(z,r,t)\,,$$

and $f = f(z,r,t)$ is any smooth test function, such that the boundary terms in the integration by parts vanish (at $r = 0$ and $r = +\infty$).

We prepare the same initial data as for the inviscid case

$$\omega(z,r,0) = \sum_{i=1}^{N} \omega_{i,\varepsilon}^0(z,r)\,,$$

with

$$\Lambda_{i,\varepsilon}(0) := \operatorname{supp} \omega_{i,\varepsilon}(\cdot,0) \subset \Sigma(\zeta^i|\varepsilon)\,,$$
$$\overline{\Sigma(\zeta^i|\varepsilon)} \subset \Pi \quad \forall\, i\,, \qquad \Sigma(\zeta^i|\varepsilon) \cap \Sigma(\zeta^j|\varepsilon) = \emptyset \quad \forall\, i \neq j\,.$$

At positive time this separation is no more true. Nevertheless, the following decomposition is possible

$$\omega_\varepsilon(z,r,t) = \sum_{i=1}^{N} \omega_{i,\varepsilon}(z,r,t)\,,$$

provided $\omega_{i,\varepsilon}(z,r,t)$, $i = 1,\dots,N$, satisfy

$$\partial_t \omega_{i,\varepsilon} + (u_z\partial_z + u_r\partial_r)\omega_{i,\varepsilon} - \frac{u_r\omega_{i,\varepsilon}}{r} = \nu\left[\partial_z^2\omega_{i,\varepsilon} + \frac{1}{r}\partial_r\left(r\partial_r\omega_{i,\varepsilon}\right) - \frac{\omega_{i,\varepsilon}}{r^2}\right],$$
$$\omega_\varepsilon(z,r,0) = \omega_{i,\varepsilon}^0(z,r)\,,$$

where (u_z,u_r) is the velocity field associated to the whole vorticity $\omega_\varepsilon(z,r,t)$.

Theorem

With an initial datum $\omega(z,r,0)$ specified above, and defining

$$\zeta^i(t) := \zeta^i + \frac{a_i}{4\pi r_i}\begin{pmatrix}1\\0\end{pmatrix} t\,, \quad i = 1,\dots,N\,,$$

then, for any $T > 0$ the following holds true. For any ε small enough and $\nu \le \varepsilon^2 |\log\varepsilon|^\gamma$, with $\gamma \in (0,1)$, there are $\zeta^{i,\varepsilon}(t) \in \Pi$, $i = 1,\dots,N$, and $R_\varepsilon > 0$ such that $\forall\, t \in [0,T]$

$$\lim_{\varepsilon\to 0} |\log\varepsilon| \int_{\Sigma(\zeta^{i,\varepsilon}(t)|R_\varepsilon)} \mathrm{d}z\,\mathrm{d}r\,\omega_{i,\varepsilon}(z,r,t) = a_i \quad \forall\, i = 1,\dots,N\,, \tag{1}$$

with

$$\lim_{\varepsilon\to 0} R_\varepsilon = 0, \qquad \lim_{\varepsilon\to 0} \zeta^{i,\varepsilon}(t) = \zeta^i(t) \quad \forall\, t \in [0,T]\,,$$

where $\omega(z,r,t)$ is the time evolution of $\omega(z,r,0)$ via the Navier-Stokes equations.

We remark that the quite strong assumption $\nu \le \varepsilon^2 |\log \varepsilon|^\gamma$ is needed to control the energy dissipated by viscosity.
This seems an unavoidable condition to obtain the concentration result (1), which requires a very small time variation of the energy.
In fact from the well known formula (in absence of external field)

$$\dot{E}(t) = -\nu \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{\xi} \sum_{i,j=1}^{3} \left(\partial_{\xi_j} u_i(\boldsymbol{\xi}, t)\right)^2 = -\nu \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{\xi}\, |\boldsymbol{\omega}(\boldsymbol{\xi}, t)|^2$$

$$= -2\pi\nu \int \mathrm{d}x\, \omega_\varepsilon^2(x, t) x_2 \ge -2\pi\nu M_2(t) \left\| \frac{\omega_\varepsilon(t)}{x_2} \right\|_{L_\infty} \ge -\frac{C\nu}{\varepsilon^2 |\log \varepsilon|^2}$$

therefore,

$$-C \frac{|\log \varepsilon|^\gamma}{|\log \varepsilon|^2} \le -\frac{C\nu}{\varepsilon^2 |\log \varepsilon|^2} \le \dot{E} \le 0\,,$$

by the assumption $\nu \le \varepsilon^2 |\log \varepsilon|^\gamma$, where $\gamma < 1$.

As for the Euler case, $\omega_{i,\varepsilon}(z, r, t)$ can be viewed as the evolution of a single vortex ring driven by the sum of the velocity generated by $\omega_{i,\varepsilon}$ itself plus an external time-depending field (the sum of the velocities generated by $\omega_{j,\varepsilon}$ for $j \neq i$).

Therefore, the proof of the Theorem will be obtained as a corollary (based on a bootstrap argument) of the analogous result for a modified system, which describes the motion of a single vortex in an external field.

A single vortex in an external field

We introduce an external time-dependent field $F^\varepsilon(x,t) = (F_1^\varepsilon(x,t), F_2^\varepsilon(x,t))$. There are $M > 0$, $a > 0$, and $\zeta^0 = (z_0, r_0)$, with $r_0 > 0$, such that

$$0 \le \omega_\varepsilon(x,0) \le \frac{M}{\varepsilon^2 |\log \varepsilon|} \quad \forall\, x \in \mathbb{R}^2 , \qquad |\log \varepsilon| \int \mathrm{d}y\, \omega_\varepsilon(y,0) = a ,$$

and

$$\Lambda_\varepsilon(0) := \operatorname{supp} \omega_\varepsilon(\cdot,0) \subset \Sigma(\zeta^0|\varepsilon) .$$

The equation of motion is obtained replacing u by $u + F^\varepsilon$, i.e.,

$$\partial_t \omega_\varepsilon + \big[(u + F^\varepsilon)\cdot\nabla\big]\omega_\varepsilon - \frac{u_2 \omega_\varepsilon}{x_2} = \nu \left[\partial_{x_1}^2 \omega_\varepsilon + \frac{1}{x_2}\partial_{x_2}(x_2 \partial_{x_2}\omega_\varepsilon) - \frac{\omega_\varepsilon}{x_2^2}\right] .$$

Concerning F^ε, we suppose that it is the sum of three terms,

$$F^\varepsilon(x,t) = \widehat{F}^\varepsilon(x,t) + \widetilde{F}^\varepsilon(x,t) + \bar{F}^\varepsilon(x,t)\,,$$

each one being a smooth time-dependent field vanishing at infinity and satisfying the conditions

$$\partial_{x_1}(x_2\widehat{F}_1^\varepsilon) + \partial_{x_2}(x_2\widehat{F}_2^\varepsilon) = 0\,, \quad \partial_{x_1}(x_2\widetilde{F}_1^\varepsilon) + \partial_{x_2}(x_2\widetilde{F}_2^\varepsilon) = 0\,,$$
$$\partial_{x_1}(x_2\bar{F}_1^\varepsilon) + \partial_{x_2}(x_2\bar{F}_2^\varepsilon) = 0\,,$$

which express the divergence free condition.

(i) there is $\widehat{r} > 0$ such that

$$\operatorname{supp} \widehat{F}^\varepsilon(\cdot, t) \subset \{x \in \mathbb{R}^2 \colon |x_2 - r_0| > \widehat{r}\} \quad \forall\, t \geq 0\,,$$

and there is $\widehat{C} > 0$ such that, for any $(x, t) \in \mathbb{R}^2 \times [0, +\infty)$,

$$|\widehat{F}^\varepsilon(x, t)| \leq \frac{\widehat{C}}{\varepsilon |\log \varepsilon|}\,;$$

without loss of generality, for later convenience, we also suppose

$$\widehat{r} < \frac{r_0}{4}\,.$$

(ii) there is $\widetilde{C} > 0$ such that, for any $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^2 \times [0, +\infty)$,

$$|\widetilde{F}^\varepsilon(x, t)| \le \frac{\widetilde{C}}{|\log \varepsilon|}, \quad |\widetilde{F}^\varepsilon(x, t) - \widetilde{F}^\varepsilon(y, t)| \le \frac{\widetilde{C}}{|\log \varepsilon|}|x - y|\,;$$

(iii) there are $\bar{C} > 0$ and $\beta > 1$ such that, for any $(x, t) \in \mathbb{R}^2 \times [0, +\infty)$,

$$|\bar{F}^\varepsilon(x, t)| \le \bar{C}\varepsilon^\beta \quad \forall\,(x, t) \in \mathbb{R}^2 \times [0, +\infty)\,.$$

The term in (i) simulates the velocity field produced by the other vortexes when the argument x of $\widehat{F}^{\varepsilon}(x, t)$ is near the center of vorticity of one of them. In this case the velocity field can diverge, when $\varepsilon \to 0$. But around the same point x the vorticity mass of $\omega_{\varepsilon}(x, t)$ is very small, by the condition

$$|r_i - r_j| \geq 2D \qquad \forall\, i \neq j$$

and this makes the integral

$$\int \mathrm{d}x\, \widehat{F}^{\varepsilon}(x, t)\, \omega_{\varepsilon}(x, t)$$

very small.

The term in (ii), present also in the Euler case, simulates the velocity field produced by the *core* of the other vortexes when the argument x of $\widetilde{F}^{\varepsilon}(x, t)$ is near the center of vorticity of $\omega_{\varepsilon}(x, t)$, and hence it is bounded and Lipschitz.

The term in (iii) simulates the velocity field produced by the *tail* of the other vortexes, and hence it is very small.

Theorem

Define

$$\zeta(t) := \zeta^0 + \frac{a}{4\pi r_0}\begin{pmatrix}1\\0\end{pmatrix} t\,.$$

Then, for any $T > 0$ *the following holds true. For any* ε *small enough and* $\nu \le \varepsilon^2 |\log \varepsilon|^\gamma$, *with* $\gamma \in (0,1)$, *there are* $\zeta_\varepsilon(t) \in \Pi$ *and* $\varrho_\varepsilon > 0$ *such that, for any* $t \in [0, T]$,

$$\lim_{\varepsilon\to 0} |\log \varepsilon| \int_{\Sigma(\zeta_\varepsilon(t)|\varrho_\varepsilon)} \mathrm{d}z\, \omega_\varepsilon(x,t) = a\,,$$

with

$$\lim_{\varepsilon\to 0} \varrho_\varepsilon = 0, \qquad \lim_{\varepsilon\to 0} \zeta_\varepsilon(t) = \zeta(t)\,,$$

where $\omega_\varepsilon(x,t)$ *is the time evolution of* $\omega_\varepsilon(x,0)$.

The proof will rely, as for the Euler case, on an a priori estimate of the *axial moment of inertia*.

Then we we show that, for any ε small enough, the vorticity mass remains highly concentrated in a thin horizontal strip centered around $x_2 = r_0$.

We will need to extend to the present context the "concentration result" already used in the inviscid case, which states that large part of the vorticity remains confined in a disk whose size is infinitesimal as $\varepsilon \to 0$.

It remains to characterize the motion of the center of vorticity in the x_1-direction, with constant speed $a/(4\pi r_0)$ when $\varepsilon \to 0$.

Since we use the following weak formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}x\,\omega_\varepsilon(x,t)f(x,t) = \int \mathrm{d}x\,\omega_\varepsilon(x,t)\big[(u+F^\varepsilon)\cdot\nabla f + \partial_t f\big](x,t) + \nu\int \mathrm{d}x\,\omega_\varepsilon(x,t)\left(\Delta f - \frac{1}{x_2}\partial_{x_2} f\right)(x,t)\,,$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, we have the technical problem to avoid the (unessential) singularity at $x_2 = 0$. We introduce then the following cut-offed versions of $B_\varepsilon(t)$ and $I_\varepsilon(t)$,

$$B_\varepsilon^*(t) = |\log\varepsilon| \int \mathrm{d}x\,\omega_\varepsilon(x,t)\,G(x_2)\,x$$

$$I_\varepsilon^*(t) = \int \mathrm{d}x\,\omega_\varepsilon(x,t)\,\big(x_2 - B_{\varepsilon,2}^*(t)\big)^2\,G(x_2)$$

where

$$G \in C^\infty(\mathbb{R};[0,1]) \quad \text{is such that} \quad G(x_2) = \begin{cases} 1 & \text{if} \quad x_2 \ge r_0 - \widehat{r}\,, \\ 0 & \text{if} \quad x_2 \le (r_0 - \widehat{r})/2\,. \end{cases}$$

We have for them the same kind of estimates obtained in the Euler case

Lemma

Under the initial conditions stated before we have

$$|\dot{B}^*_{\varepsilon,2}(t)| \leq \frac{C}{|\log \varepsilon|} \quad \forall\, t \in [0,\, T_\varepsilon]$$

and

$$I^*_\varepsilon(t) \leq \frac{C}{|\log \varepsilon|^2} \quad \forall\, t \in [0,\, T_\varepsilon]\,.$$

With these estimates we are able to implement an iterative scheme as in the Euler case, showing that the vorticity mass out of a thin strip parallel to the x_1-axis and centered around $x_2 = r_0$ is very small (less than any power of ε, for $\varepsilon \to 0$).

With respect to the analogous result in the Euler case, here we have to take account of the presence of the viscosity (in particular, this implies that the vorticity is not compactly supported) and the presence of a less regular external field. However, these differences do not condition too much the reasoning.

Lemma

Let m_t be defined as

$$m_t(h) = \int_{|x_2 - B^*_{\varepsilon,2}(t)| > h} \mathrm{d}x\, \omega_\varepsilon(x,t)\,.$$

Then for each $\ell > 0$ and $k \in \left(0, \frac{1}{4}\right)$,

$$\lim_{\varepsilon \to 0} \max_{t \in [0, T_\varepsilon]} \varepsilon^{-\ell} m_t\left(\frac{1}{|\log \varepsilon|^k}\right) = 0\,.$$

Analysis of the axial motion

We need first a concentration result, already used in the Euler case, in the present one with a small viscosity.

Lemma

Taking $\varepsilon_0 \in (0,1)$*, for any* $\eta \in (\gamma, 1)$ *there are* $\varepsilon_1 \in (0, \varepsilon_0)$, $\varrho_1 > 0$, $C_1 > 0$*, and* $q_\varepsilon(t) \in \mathbb{R}^2$*, such that*

$$|\log \varepsilon| \int_{\Sigma(q_\varepsilon(t), \varrho_\varepsilon)} \mathrm{d}x \, \omega_\varepsilon(x,t) \geq 1 - \frac{C_1}{|\log \varepsilon|^{\eta - \gamma}} \, ,$$

$\forall\, t \in [0, T]$, $\forall\, \varepsilon \in (0, \varepsilon_1)$*, with* $\varrho_\varepsilon = \varrho_1 \varepsilon \exp(|\log \varepsilon|^\eta)$.

This inequality is deduced from an upper bound on the kinetic energy functional $E = \frac{1}{2}\int \mathrm{d}\boldsymbol{\xi}\, |\boldsymbol{u}(\boldsymbol{\xi}, t)|^2$, which in cylindrical coordinates $x = (x_1, x_2) = (z, r)$ takes the form

$$E(t) = \frac{1}{2}\int \mathrm{d}x\, 2\pi x_2 |u(x,t)|^2\,,$$

combined with the bounds

$$M_0(t) = \int \mathrm{d}x\, \omega_\varepsilon(x,t) \le \frac{C}{|\log \varepsilon|}\,,$$
$$M_2(t) = \int \mathrm{d}x\, x_2^2 \omega_\varepsilon(x,t) \le \frac{C}{|\log \varepsilon|}\,,$$
$$E(t) \ge E(0) - \frac{C\nu}{\varepsilon^2 |\log \varepsilon|^2} \ge E(0) - \frac{C}{|\log \varepsilon|^{2-\gamma}}\,,$$

with $\gamma \in (0,1)$ (for $t \in [0, T]$ and any ε small enough). These bounds are due, in the present situation, to the effect of viscosity *and* the external field (which simulates the action of the other $N-1$ vortex rings).

It remains to show that

$$\lim_{\varepsilon\to 0} q_{\varepsilon,2}(t) = r_0\,, \quad \lim_{\varepsilon\to 0} q_{\varepsilon,1}(t) = z_0 + \frac{t}{4\pi r_0} \quad \forall\, t \in [0, T]\,.$$

The first limit is quite obvious, since the vorticity mass in the region

$$|x_2 - r_0| > \frac{1}{|\log\varepsilon|^k}$$

is smaller than ε^ℓ.

The second limit can be deduced by the following Lemma

Lemma

Let $B_{\varepsilon,1}(t)$ be the first component of the center of vorticity and let $J_\varepsilon(t)$ be the corresponding radial moment of inertia,

$$J_\varepsilon(t) = \int \mathrm{d}x\, \omega_\varepsilon(x,t)\,(x_1 - B_{\varepsilon,1}(t))^2 \ .$$

Then,

$$\lim_{\varepsilon\to 0} B_{\varepsilon,1}(t) = z_0 + \frac{t}{4\pi r_0} \quad \forall\, t \in [0,\, T]\,,$$
$$J_\varepsilon(t) \le \frac{C}{|\log\varepsilon|} \quad \forall\, t \in [0,\, T]\,.$$

Indeed, we claim that the previous Lemma implies

$$\lim_{\varepsilon \to 0} |B_{\varepsilon,1}(t) - q_{\varepsilon,1}(t)| = 0 \quad \forall\, t \in [0, T]\,,$$

which gives the limit for $q_{\varepsilon,1}(t)$ as $\varepsilon \to 0$. Note that also

$$\lim_{\varepsilon \to 0} |B_{\varepsilon,2}(t) - q_{\varepsilon,2}(t)| = 0 \quad \forall\, t \in [0, T]\,.$$

The first limit is obtained by using $J_\varepsilon(t)$, the Concentration Lemma, and Cauchy-Schwarz inequality

$$|B_{\varepsilon,1}(t) - q_{\varepsilon,1}(t)| \le C\varrho_\varepsilon + \sqrt{\frac{C}{|\log \varepsilon|^{\eta-\gamma}}}\sqrt{|\log \varepsilon| J_\varepsilon(t)}\,.$$

Thank you!