On maximally mixed equilibria of 2D perfect fluids

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Outline

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The 2D Euler equation

$$\partial_t \omega + \boldsymbol{u} \cdot \nabla \omega = 0, \qquad \text{in } M \subset \mathbb{R}^2 \text{ bounded (simply connected)},$$
$$\boldsymbol{u} = \nabla^\perp \psi, \quad \Delta \psi = \omega,$$
$$\omega|_{t=0} = \omega_0, \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0.$$

Given $\omega_0 \in L^\infty(M)$, we have

$$\omega(t) = \omega_0 \circ \Phi_t^{-1}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t = \boldsymbol{u}(\Phi_t, t), \quad \Phi_0 = \mathrm{id},$$

where Φ_t is a **volume preserving homeomorphism**.

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where Φ_t is a **volume preserving homeomorphism**.

- Yudovich '63: existence (Wolibner '33) and uniqueness of weak solutions on $X = \{\omega \in L^\infty(M) : \|\omega\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}\}$.
- $\omega(t_j) \overset{*}{\rightharpoonup} \bar{\omega}$ along subsequence $t_j \to \infty$ for some $\bar{\omega}$, namely

$$\int_M g(x)\omega(t_j, x)\mathrm{d}x \to \int_M g(x)\bar{\omega}(x)\mathrm{d}x, \quad \text{for any } g \in L^1(M).$$

Robust invariants: Kinetic energy

$$\mathsf{E}(\omega) := \frac{1}{2}\int_M |\boldsymbol{u}|^2 \mathrm{d}x = -\frac{1}{2}\int_M \omega\psi \mathrm{d}x, \qquad \Delta\psi = \omega,$$

$$\mathsf{E}(\omega(t)) = \mathsf{E}(\omega_0), \qquad \text{and} \qquad \mathsf{E}(\omega_0) = \lim_{t_j\to\infty} \mathsf{E}(\omega(t_j)) = \mathsf{E}(\bar{\omega})$$

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Fragile invariants: by the transport structure, $\omega(t) = \omega_0 \circ \Phi_t^{-1}$ we have

$$\text{Casimirs}: \quad \mathcal{C}_f(\omega(t)) := \int_M f(\omega(t))\mathrm{d}x = \int_M f(\omega_0 \circ \Phi_t^{-1})\mathrm{d}x = \mathcal{C}_f(\omega_0), \text{ for } t < \infty.$$

In general, $\mathcal{C}_f(\omega_0) = \lim_{t_j\to\infty} \mathcal{C}_f(\omega(t_j)) \neq \mathcal{C}_f(\bar{\omega})$

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In general, $\mathcal{C}_f(\omega_0) = \lim_{t_j\to\infty} \mathcal{C}_f(\omega(t_j)) \neq \mathcal{C}_f(\bar\omega)$
If f is convex, by l.s.c. we have

$$\mathcal{C}_f(\bar\omega) \le \liminf_{t_j\to\infty} \mathcal{C}_f(\omega(t_j)) = \mathcal{C}_f(\omega_0).$$

Mixing: $\mathcal{C}_f(\bar\omega) < \mathcal{C}_f(\omega_0)$.
What can be said at $t = \infty$ accounting only for the invariants?

Example

- Let $M = \mathbb{T}^2$. Then $\omega(x) = \sum_{k \in \mathbb{Z}^2} \hat{\omega}_k e^{ik \cdot x}$.
- Suppose that $\bar{\omega}$ is s.t. $\widehat{\bar{\omega}}_k = 0$ for $|k| < N$.

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- Since $\|\bar{\omega}\|_{L^2} \leq \|\omega_0\|_{L^2}$, if $N \gg 1$, we get

$$\mathrm{E}(\bar{\omega}) = \frac{1}{(2\pi)^2} \sum_{|k|\geq N} \frac{1}{|k|^2} |\widehat{\bar{\omega}}_k|^2 \lesssim \frac{1}{|N|^2} \|\bar{\omega}\|_{L^2}^2 \lesssim \frac{1}{|N|^2} \|\omega_0\|_{L^2}^2 < \mathrm{E}(\omega_0).$$

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Some low frequency piece must survive. Large spatial scale structures!
Inverse cascade, Kraichnan '67.

If $\hat{\omega}_k = 0$ for all $|k| > N$, then ω is stationary (Elgindi/Hu/Šverák '15).
Something which is not steady must always have high frequency part.

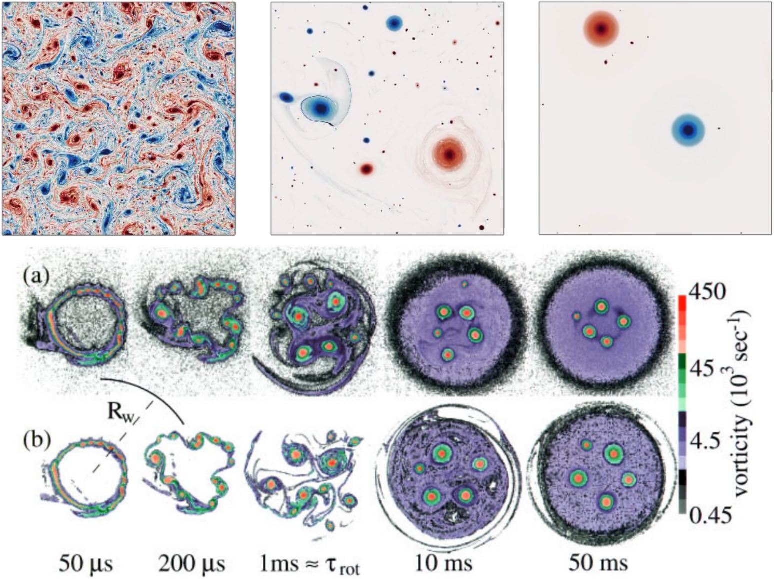
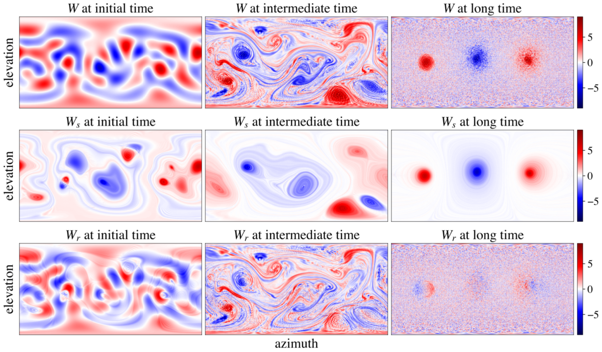


Figure: **Top**: Numerics by Constantinou/Drivas '21. **Bottom**: Schecter/Dubin /Fine/Driscoll '99 (a) experiment of pure electron plasma (b) numerics of 2D Euler.



Modin/Viviani '21 numerics on Zeitlin model. Nice idea: split the vorticity $\omega = \omega_s + \omega_r$ with ω_s average along the streamlines $\gamma(h) = \{x : \psi(x) = h\}$

$$\omega_s(h) = \int_{\gamma(h)} \frac{\omega}{|\nabla\psi|} \mathrm{d}\ell$$

- ω_s is large scale, contains most of the energy but only a fraction of enstrophy.
- ω_r is small scale, very little energy but more enstrophy w.r.t ω_s.

SUPPLEMENTO AL VOLUME VI, SERIE IX DEL NUOVO CIMENTO N. 2, 1949

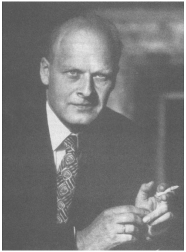
XIII.

Statistical Hydrodynamics. (*)

L. ONSAGER
New Haven, Conn.

Ergodic Motion of Parallel Vortices.

The formation of large, isolated vortices is an extremely common, yet spectacular phenomenon in unsteady flow. Its ubiquity suggests an explanation on statistical grounds.



Statistical mechanics approach for N-point vortices. Clustering of likely signed vortices and negative "temperature" states. But $[N \to \infty, T \to \infty] \neq 0$. See Eyink/Sreenivasan '06.

Statistical hydrodynamics after Onsager

- Kraichnan '67, Joyce/Montgomery '70, Caglioti/Lions/Marchioro/Pulvirenti, Chavanis, Eyink/Spohn '90s...
  See review of Bouchet/Venaille '12.
  For example, Kraichnan predicts that the most probable state minimizes the enstrophy in $X \cap \{\mathrm{E} = \mathrm{E}_0\}$. Dynamically accessible?
- Miller, Robert, Sommeria '90s: Finite dimensional approximation of 2D Euler by discretizing vorticity. Stat. mech. approach where permutation of squares generates the accessible states. They obtain a variational problem taking into account **all** finite time conserved quantities.
- Turkington '99: Fourier truncation of 2D Euler in $\mathbb{T}^2$ with the Fejèr kernels:

$$F_N(x) = (2\pi)^{-2} \sum_{|k| \leq N} (1 - |k_1|/N)(1 - |k_2|/N)\mathrm{e}^{ik\cdot x}$$

$K_N[\omega] := F_N * \omega$.
This kernel has some special properties.

Two conjectures about the infinite time-behaviour

Statements from Drivas/Elgindi '22.

Šverák's conjecture '11: Generic initial data $\omega_0 \in L^\infty(M)$ gives rise to inviscid incompressible motions whose vorticity orbits $\{\omega(t)\}_{t\in\mathbb{R}}$ are not precompact in $L^2(M)$.

Some **mixing**, e.g. $\|\overline{\omega}\|_{L^2} < \|\omega_0\|_{L^2}$, is happening.
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> **Shnirelman's conjecture '13:** For any initial data $\omega_0 \in L^\infty(M)$, the collection of $L^2(M)$ weak limits of the orbits $\{\omega(t)\}_{t\in\mathbb{R}}$ consists of vorticities whose orbits are precompact in $L^2(M)$.

After $t = \infty$, you cannot mix more. Euler is "maximally mixing".
This is consistent with predictions of statistical hydro of reaching a stationary state. It is also true if we reach a periodic or quasi periodic state.

See Bedrossian/Masmoudi '13 (and related works) and Elgindi/Murray/A.R. Said '22 A.R. Said/Jeong '23.

Outline

Area preserving rearrangements and weak-* closure

For 2D Euler, $\omega(t) = \omega_0 \circ \Phi_t^{-1}$ is a rearrangement of ω_0. Define

$$\mathcal{O}_{\omega_0} = \{\omega_0 \circ \varphi \, : \, \varphi \text{ area preserving diffeo of } M\}.$$

- Consider the weak-* closure of this set.
  Namely, for $\omega_0 \in L^\infty$, let $\{\varphi_n\}_{n=1}^\infty$. Then

$$\omega_0 \circ \varphi_n \overset{*}{\rightharpoonup} \bar{\omega} \in \overline{\mathcal{O}_{\omega_0}}^{*} \qquad \text{(also } \omega_0 \circ \varphi_n \overset{L^2}{\rightharpoonup} \bar{\omega}\text{)}$$

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- The set $\overline{\mathcal{O}_{\omega_0}}^*$ is convex and can be characterized (Chong, Ryff... '60-'70s)

$$\overline{\mathcal{O}_{\omega_0}}^* = \left\{\omega \in X,\ \int_M \omega = \int_M \omega_0,\ \int_M f(\omega) \leq \int_M f(\omega_0) \text{ for all convex } f\right\},$$

where $X = \{\|\omega\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}\}$.

Chong, Ryff, 60s-70s and Brenier/Gangbo '03 observed also that

$$\overline{\mathcal{O}_{\omega_0}}^* = \{\omega \in X \,:\, \omega = K[\omega_0],\, K \text{ bistochastic}\}$$

where K is a **bistochastic operator** if:

- $K : M \times M \to \mathbb{R}$ is such that

$$K \geq 0, \quad \int_M K(x, \cdot)\mathrm{d}x = \int_M K(\cdot, y)\mathrm{d}y = 1,$$

$$K[\omega](x) = \int_M K(x, y)\omega(y)\mathrm{d}y.$$

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Examples:

- $K_{mix} = 1/|M|$ and $K_{mix}[\omega] = \int_M \omega/|M|$. Complete mixing.
- $K_\varphi = \delta(y - \varphi(x))$ and $K_\varphi[\omega] = \omega \circ \varphi$, for any area preserving map φ.
- If K_1, K_2 bistochastic, then also $\widetilde{K} = \lambda K_1 + (1-\lambda)K_2$, $\lambda \in [0,1]$.
- $K_N[\omega] = F_N * \omega$ where F_N is the Féjer kernel (Turkington)
- $K_\gamma(x,y) = \delta(y - \gamma(x))/|\nabla\psi|$ the Modin/Viviani streamlines projection.

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$$= \{\omega \in X\ :\ \omega = K[\omega_0],\ K \text{ bistochastic}\}$$

where $X = \{\|\omega\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}\}$.

This set is too large to be relevant for Euler, no energy conservation.
If $\int \omega_0 = 0$ then $K_{mix}[\omega_0] = 0 \in \overline{\mathcal{O}_{\omega_0}}^*$ (complete mixing in $\mathbb{T}^2$ for instance)

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All the infinite time limits of 2D Euler are contained in:

$$\Omega_+(\omega_0) = \bigcap_{s\ge 0} \overline{\{\omega_0 \circ \Phi_t^{-1},\ t \ge s\}}^* \subseteq \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathrm{E}(\omega) = \mathrm{E}(\omega_0)\}.$$

Shnirelman '93: found special steady states of 2D Euler in $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathrm{E}(\omega) = \mathrm{E}(\omega_0)\}$ as a consequence of Zorn's lemma.

Outline

Minimal elements or maximally mixed states

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$$= \left\{\omega \in X,\ \int_M \omega = \int_M \omega_0,\ \int_M f(\omega) \le \int_M f(\omega_0) \text{ for all convex } f\right\}.$$

We have two natural partial ordering. Let $\omega_1, \omega_2 \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$:

(1) Shnirelman '93: $\omega_1 \preceq_S \omega_2$ if $\exists$ K bistochastic s.t. $\omega_1 = K\omega_2$.

(2) D/Drivas '22: $\omega_1 \preceq \omega_2$ if $\exists$ a strictly convex f s.t. $\mathcal{C}_f(\omega_1) \le \mathcal{C}_f(\omega_2)$.

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Minimal elements: An $\omega^* \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ is *minimal* if for all $\omega \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ s.t. $\omega \preceq \omega^*$ then $\omega^* \preceq \omega$. We say $\omega \simeq \omega^*$ (or $\preceq_S, \simeq_S$).

Minimal elements are maximally mixed.
Decrease of any Casimir $\mathcal{C}_f$ requires a change of energy.

Equivalence of the two definitions

Lemma (D/Drivas '22)

Let $\omega_2 \in X$, K bistochastic and $\omega_1 = K[\omega_2]$. There exists $\widetilde{K}$ bistochastic s.t. $\omega_2 = \widetilde{K}[\omega_1]$ if and only if $\mathcal{C}_f(\omega_1) = \mathcal{C}_f(\omega_2)$ for any strictly convex f.

- We state for **any** strictly convex f since if it is true for one it is true for all. In fact, we prove that ω_1 and ω_2 are equimeasurable.
- The two concepts of minimal flows are equivalent, $\omega_1 \simeq \omega_2 \iff \omega_1 \simeq_S \omega_2$.

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The proof is basically the following:
◇ Jensen's inequality + f strictly convex $\Longrightarrow$ equimeasurability of ω_1 and ω_2.
◇ Brenier/Gangbo '03 approximation theorem:
$\forall K$ there exists a sequence of *permutations of squares* $\{p_n\}_{n=1}^{\infty}$ s.t.

$$\lim_{n\to\infty} \int_M g(x, p_n(x))\mathrm{d}x = \iint_{M\times M} g(x,y)K(x,y)\mathrm{d}x\mathrm{d}y, \qquad \text{for all } g \in C(M\times M).$$

Using this with equimeasurability we can "invert" K.
Roughly speaking, $\widetilde{K}$ is obtained from p_n^{-1}.

Theorem (D/Drivas '22)

Let $\omega_0 \in L^\infty(M)$ *and* f *strictly convex. There exists* $\omega^* \in X$ *such that:*

(i) $\mathcal{C}_f(\omega^*) = \min\{\mathcal{C}_f(\omega) \,:\, \omega \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}\}$

(ii) ω^* *is a minimal element (maximally mixed) in* $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$

(iii) *There exists a bounded and monotone* F *such that* $\omega^* = F(\psi^*)$

(iv) *There exists* g *continuous and convex,* $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0$ *such that*
$\mathsf{J}_g(\omega^*) = \min\{\mathcal{C}_g(\omega) + \alpha\mathcal{C}_f(\omega) + \beta\mathsf{E}(\omega) + \gamma \int_M \omega \,:\, \omega \in X\}.$

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- Shnirelman '93: Minimal elements are **all** steady states. Some of them might not come from a variational problem though.
- Point (*iv*) is basically a Lagrange multiplier rule (variational problem in the uncostrained space X).
  Obtained with an abstract theorem of Rakotoson/Serre '93.
  If $\alpha \neq 0$ then F stricly monotone.
  $\alpha = 0$ includes states with constant vorticity somewhere.

Examples

- If $\omega_0 = F(\psi_0)$ is constant or Arnold stable, i.e. $0 < F'(\psi) < \infty$ or $-\lambda_1 < F'(\psi) < 0$ where λ_1 first eigenvalue of Δ, then

$$\Omega_+(\omega_0) = \overline{\mathcal{O}_{\omega_0}}^* \cap \{E = E_0\} = \{\omega_0\}.$$

Remark: Arnold stable states conform to the geometry of the domain, Constantin/Drivas/Ginsberg '20 (see also Hamel/Nadirashvili '17-'19). We show later that minimal flows need **not** inherit any domain symmetry.

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- If $\omega_0 = \sum_{i=1}^N a_i \chi_{A_i}$, $a_i < a_{i+1}$, $\cup_i A_i = M$ then

$$\overline{\mathcal{O}_{\omega_0}}^* = \left\{ \omega \in X, : \int_M \omega = \int_M \omega_0, \int_M (\omega - a_i)_+ \leq \int_M (\omega_0 - a_i)_+, \text{ for all } i \right\}.$$

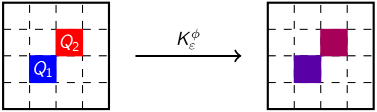
Finite number of nonsmooth inequality constraints. When $a_1 = -a_2 = 1$, $a_i = 0$ for $i \geq 3$, the only constraint is $\|\omega\|_{L^\infty} \leq 1$ (see also Šverák's lecture notes '11).

Outline

◇ Assume that ω^* is a minimizer of $\mathcal{C}_f$ on $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$.

◇ Let ϕ be a permutation of two small squares, say Q_1, Q_2, where $\omega^*|_{Q_1} \neq \omega^*|_{Q_2}$. Define:

$$K_\varepsilon^\phi \omega = (1-\varepsilon)\omega + \varepsilon(\omega \circ \phi),$$



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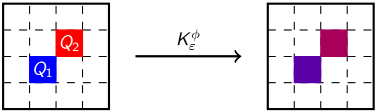
◇ Let ϕ be a permutation of two small squares, say Q_1, Q_2, where $\omega^*|_{Q_1} \neq \omega^*|_{Q_2}$.
Define:

$$K_\varepsilon^\phi \omega = (1-\varepsilon)\omega + \varepsilon(\omega \circ \phi),$$



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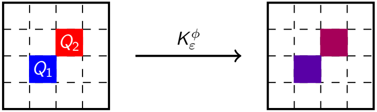

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⋄ If claim not true, $\exists$ ϕ_1, ϕ_2 s.t. $\mathsf{E}(K_\varepsilon^{\phi_1}\omega^*) > \mathsf{E}(\omega^*)$, $\mathsf{E}(K_\varepsilon^{\phi_2}\omega^*) < \mathsf{E}(\omega^*)$. Then $\exists$ $0 < \lambda < 1$ s.t. $\tilde{K}_\varepsilon = \lambda K_\varepsilon^{\phi_1} + (1-\lambda)K_\varepsilon^{\phi_2}$ satisfy $\mathsf{E}(\tilde{K}_\varepsilon \omega^*) = \mathsf{E}(\omega^*)$. But $\mathcal{C}_f(\tilde{K}_\varepsilon \omega^*) < \mathcal{C}_f(\omega^*)$ since f strictly convex. Contradiction.

Outline

Couette flow is minimal: $\omega_* = -1$ in $\mathbb{T} \times [0,1]$. Then $v_* = (x_2, 0)$.
It conforms to the domain symmetries. What happens if we perturb it?

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$$\partial_t w + x_2 \partial_{x_1} w = 0 \quad \Longrightarrow \quad w = w_0(x_1 - x_2 t, x_2) \quad \Longrightarrow \quad w(t) \xrightarrow{L^2} \int_{\mathbb{T}} w_0 \mathrm{d}x_1$$

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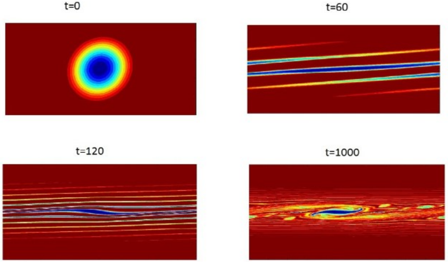


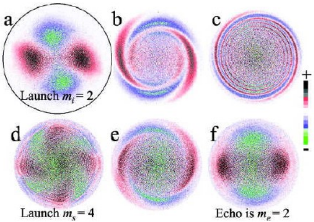


Figure: **Left**: perturbation of the Couette flow, i.e. $\boldsymbol{u} = (x_2, 0)$, Shnirelman '13.
Right: experiment by Yu/Driscoll '02 in a pure electron plasma. See also Vanneste '98

Nonlinear results

For the Couette flow:

- Lin/Zeng '10: there exists non-shear steady states s.t. $\|w_0\|_{H^{\frac{3}{2}-}} \leq \varepsilon$. Cats-eye structures. See also Castro-Lear '22.
- Bedrossian/Masmoudi '13: if w_0 small in Gevrey-2^+, i.e $\widehat{w}_{0,k} \sim \varepsilon \mathrm{e}^{-\sqrt{|k_1|+|k_2|}}$, then nonlinear invicisd damping in $\mathbb{T} \times \mathbb{R}$ holds. Inviscid relaxation to equilibrium by mixing.
- Deng/Masmoudi '18: instability of vorticity if w_0 in Gevrey-2^-. Inviscid damping?
- Ionescu/Jia '19: nonlinear inviscid damping in $\mathbb{T} \times [0, 1]$ and Gevrey-2.

Other stationary states:

- Ionescu/Jia '20: axisymmetrization around point vortex.
- Ionescu/Jia '20 and Masmoudi/Zhao '20: nonlinear inviscid damping for strictly monotone shear flows.
- Coti Zelati/Elgindi/Widmayer '20: non-shear stationary states near Kolmogorov $(\sin(x_2), 0)$ and Poiseulle $(1 - y^2, 0)$ flows.
- M. Nualart '22: non-zonal stationary states around zonal flows in the sphere.

Theorem (D/Drivas '22)

Let $M = \mathbb{T} \times [0,1]$, $\omega_b \in L^\infty(M)$ *and*

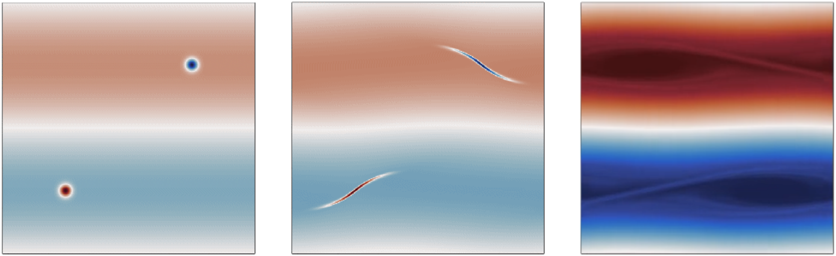
$$\mathsf{L_m}(\omega) := \int_M e_1 \cdot \boldsymbol{u}\,\mathrm{d}x = \int_{\mathbb{T}} u_1(x_1, 1)\mathrm{d}x_1 + \int_M x_2 \omega(x)\mathrm{d}x.$$

Then, for any $\delta > 0$, $\exists\, \xi \in C^\infty$ *s.t.*

$$\|\xi - \omega_b\|_{L^1} \leq \delta$$

and for which the set $\overline{\mathcal{O}_\xi}^* \cap \{\mathsf{E} = \mathsf{E}(\xi)\} \cap \{\mathsf{L_m} = \mathsf{L_m}(\xi)\}$ *contains no shear flows.*

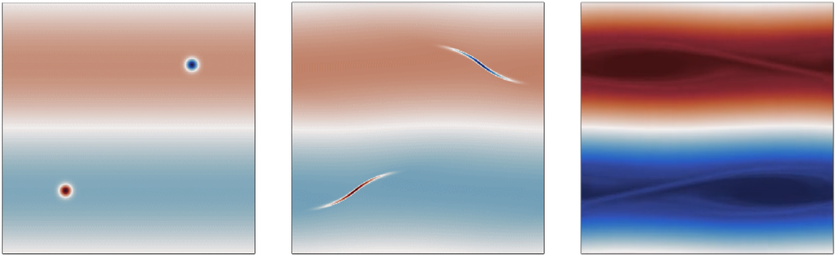
- Starting from the field ξ, we can construct non-shear minimal flows (hence steady) with our variational problem.
- In particular, $\Omega_+(\xi)$ does not contain shear flows. Choosing $\omega_b = -v(y)$, we show that convergence back to a shear is not possible for ξ.
- The field ξ is explicit and **not** steady (details later).
- Analogous result if M is an annulus. In the disk?



In this case $\omega_b = \sin(y)$. We choose ξ as highly peaked vortices embedded in the background. Namely,

$$\|\xi - \omega_b\|_{L^\infty} \approx \varepsilon^{-2}, \qquad |\mathrm{supp}(\xi - \omega_b)| \lesssim \varepsilon^2,$$
$$\mathsf{E}(\xi) - \mathsf{E}(\omega_b) \approx \delta^2 |\log(\varepsilon)|, \qquad |\mathsf{L}_\mathsf{m}(\xi) - \mathsf{L}_\mathsf{m}(\omega_b)| \lesssim \delta.$$

We are perturbing the background ω_b with an approximation of point vortices.



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Main idea: shear flows are $1D$ objects for which the Biot-Savart law is non-singular, so energy $O(1)$. Approximated point vortices exploit the singularity of Biot-Savart in $2D$, so energy $O(|\log(\varepsilon)|)$. They cannot be sheared out due to energy conservation and control on L^1 norms.

◇ Choose

$$\xi = \omega_b + \delta\varepsilon^{-2}\mathbb{1}_{A_\varepsilon}(x), \qquad |A_\varepsilon| = \varepsilon^2.$$

◇ Compute that

$$\|\xi - \omega_b\|_{L^1} \lesssim \delta, \quad E(\xi) \approx E(\xi - \omega_b) \approx \delta^2|\log(\varepsilon)|$$

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Using the characterizations of $\overline{\mathcal{O}_\xi}^*$, worst case scenario is

$$\widetilde{\omega}_{\mathsf{s}}(x_2) \approx \begin{cases} O(\mu^{-2}) & \text{on } \widetilde{A}_{\mu^2},\ |\widetilde{A}_{\mu^2}| \leq \mu^2, \\ O(1) & \text{on } [0,1] \setminus A_{\mu^2}, \end{cases} \qquad 0 < \mu \ll 1.$$

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This lead to a contradiction, namely, let G be the Green's function on $\mathbb{T} \times [0,1]$

$$\mathsf{E}[\widetilde{\omega}_{\mathsf{s}}] = -\frac{1}{2}\iint_{M\times M} G(x,y)\widetilde{\omega}_{\mathsf{s}}(x_2)\widetilde{\omega}_{\mathsf{s}}(y_2)\mathrm{d}x\mathrm{d}y = O(1) \ll \delta^2|\log(\varepsilon)| \approx \mathsf{E}[\xi].$$

Remark: Conservation of momentum is crucial to avoid shear flows s.t.
$\boldsymbol{u} = (v(x_2) + |\log(\varepsilon)|, 0)$. Vorticity invisible to such shift but not momentum.

A future direction

Obtain minimal flows dynamically. With Drivas and Galeati we consider

$$\partial_t \omega + u \cdot \nabla \omega = \kappa u \cdot \nabla (u \cdot \nabla \omega)$$
$$u = \nabla^\perp \psi, \qquad \Delta \psi = \omega.$$

This is related to *anticipated vorticity model* Sadourny/Basdevant '80s, recently considered by Gay-Balmaz/Holm '12 as well.

- The energy is conserved, namely $\frac{\mathrm{d}}{\mathrm{d}t} \int_M \omega \psi = 0$.
- $\omega = F(\psi)$ are steady states for this model.
- Strictly convex Casimirs are "dissipated"

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M f(\omega) \mathrm{d}x + \kappa \left\| \sqrt{f''(\omega)} (u \cdot \nabla \omega) \right\|_{L^2}^2 = 0$$

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- Yudovich type theory (global well-posedness in L^∞)?
- Long time-behavior around particular steady states?