How to analyze, model and compute turbulent flows using wavelets

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WHAT IS TURBULENCE ?

WHAT IS WAVELET REPRESENTATION ?

HOW TO ANALYZE TURBULENT FLOWS ?

HOW TO MODEL AND COMPUTE TURBULENT FLOWS ?

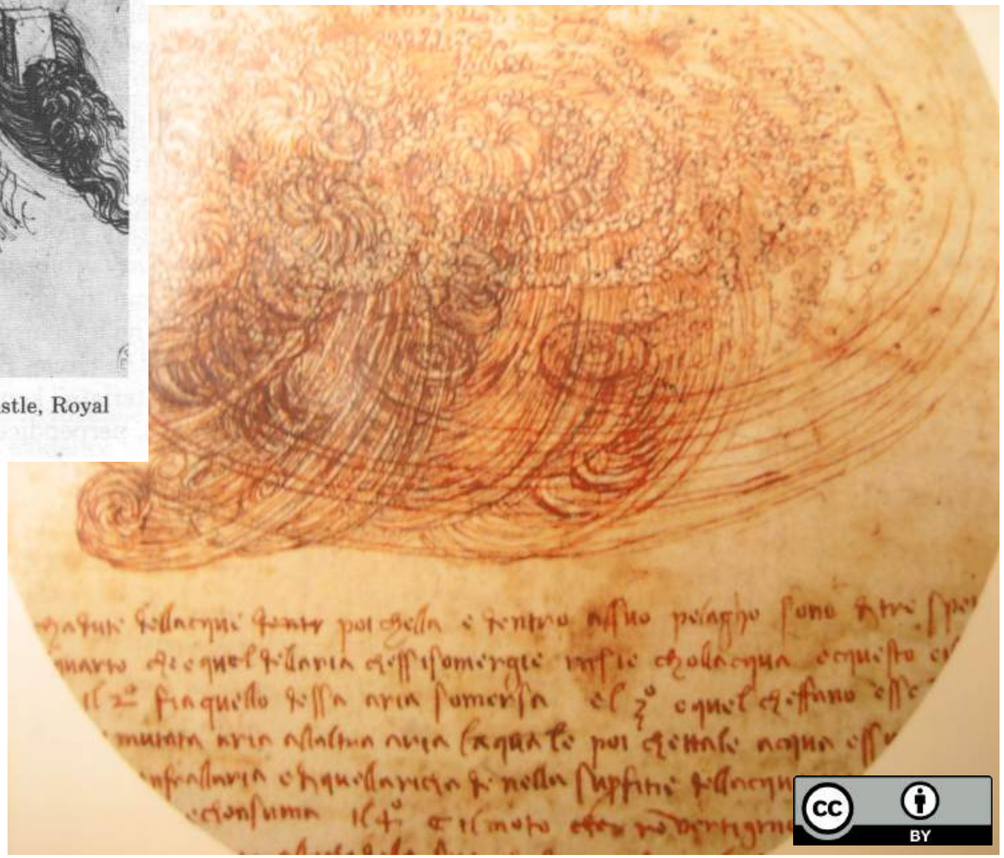
CONCLUSION

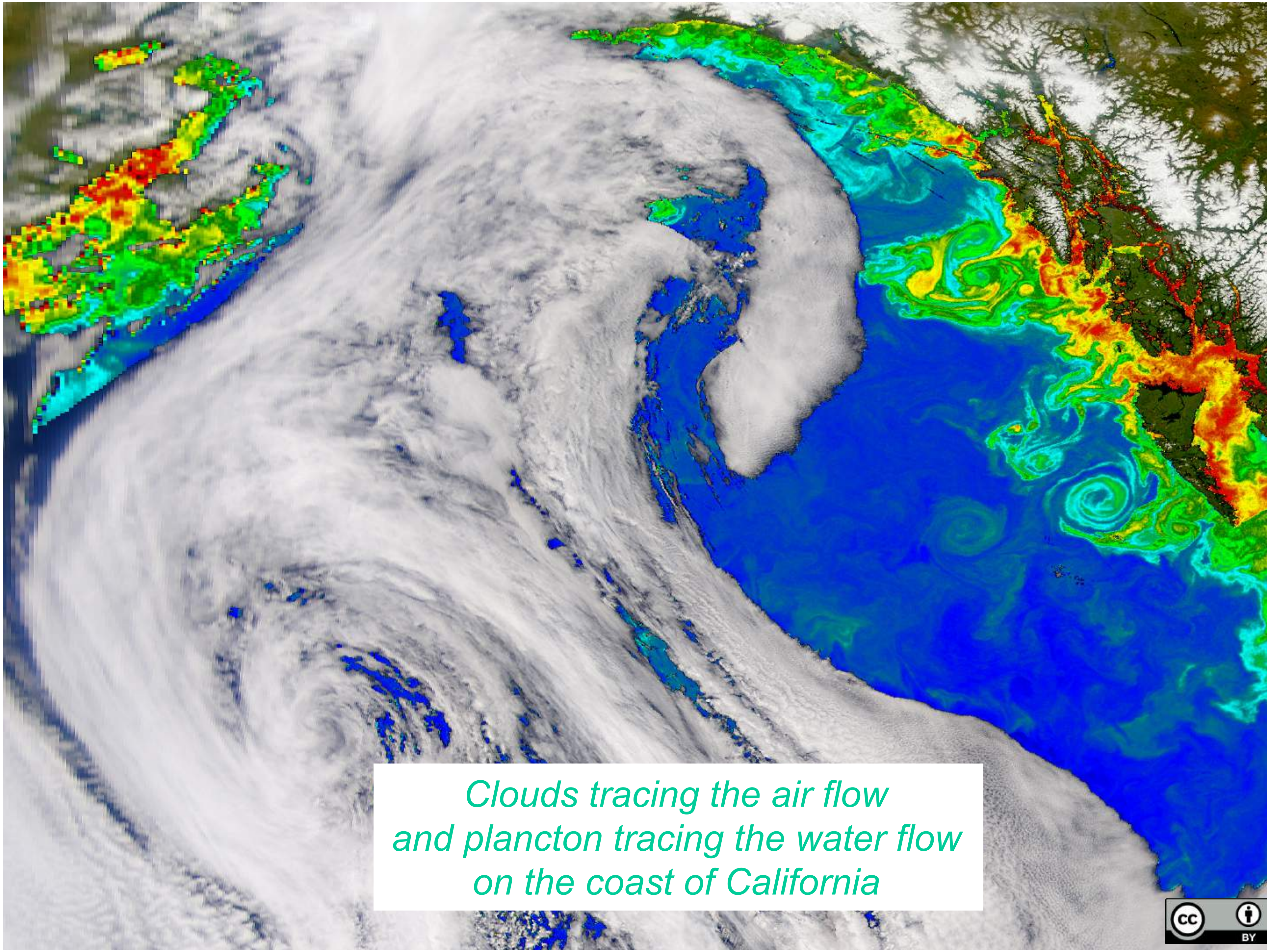
Observe - Question - Represent

Figure 1.8. Leonardo: *Old man and Vortices;* probably a self-portrait (Windsor Castle, Royal Library, copyright reserved).

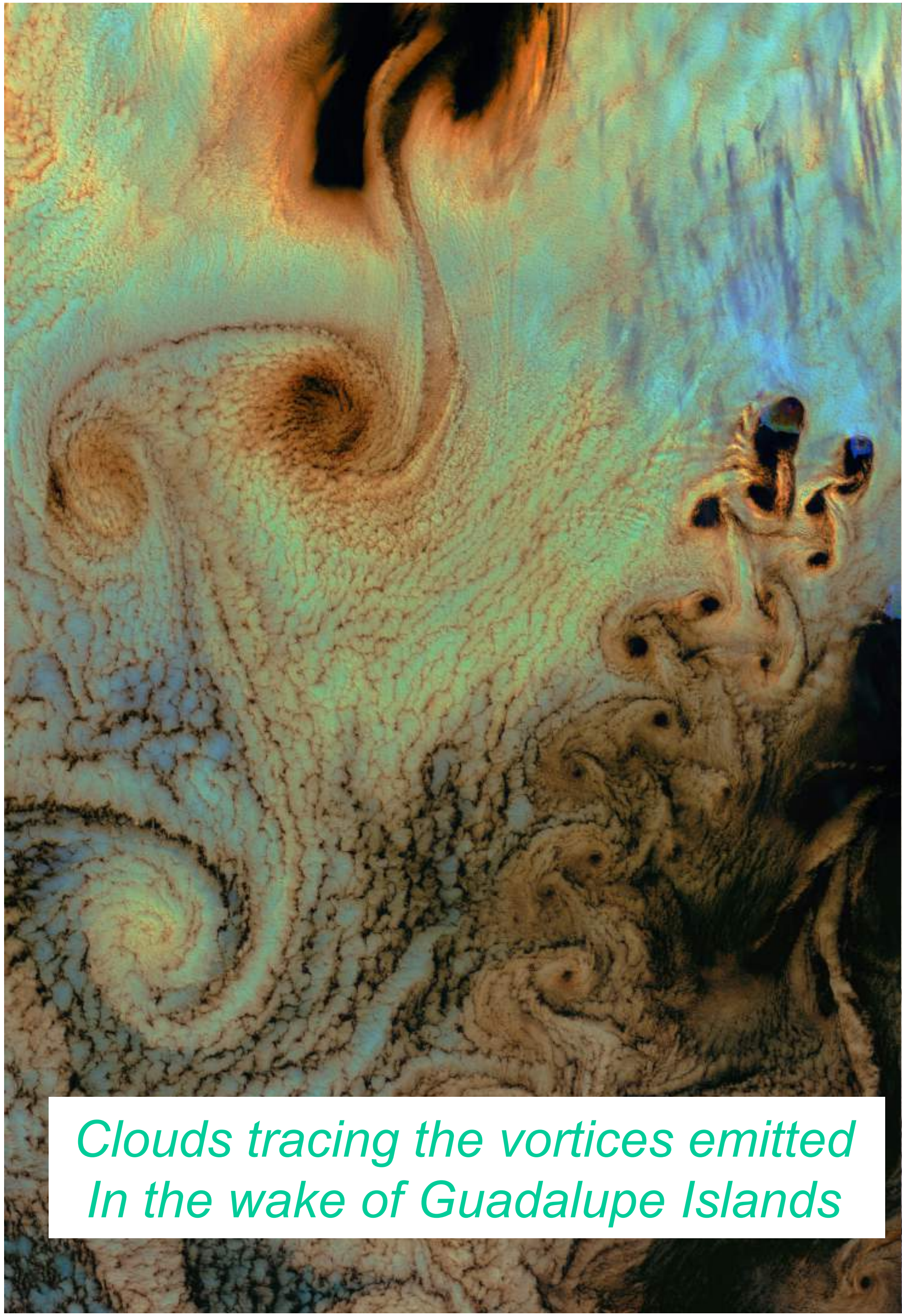
Leonardo da Vinci
(1452-1519)
introduced the word
turbolenza

Self-portrait :
`Old man and vortices'
Windsor Castle Collection

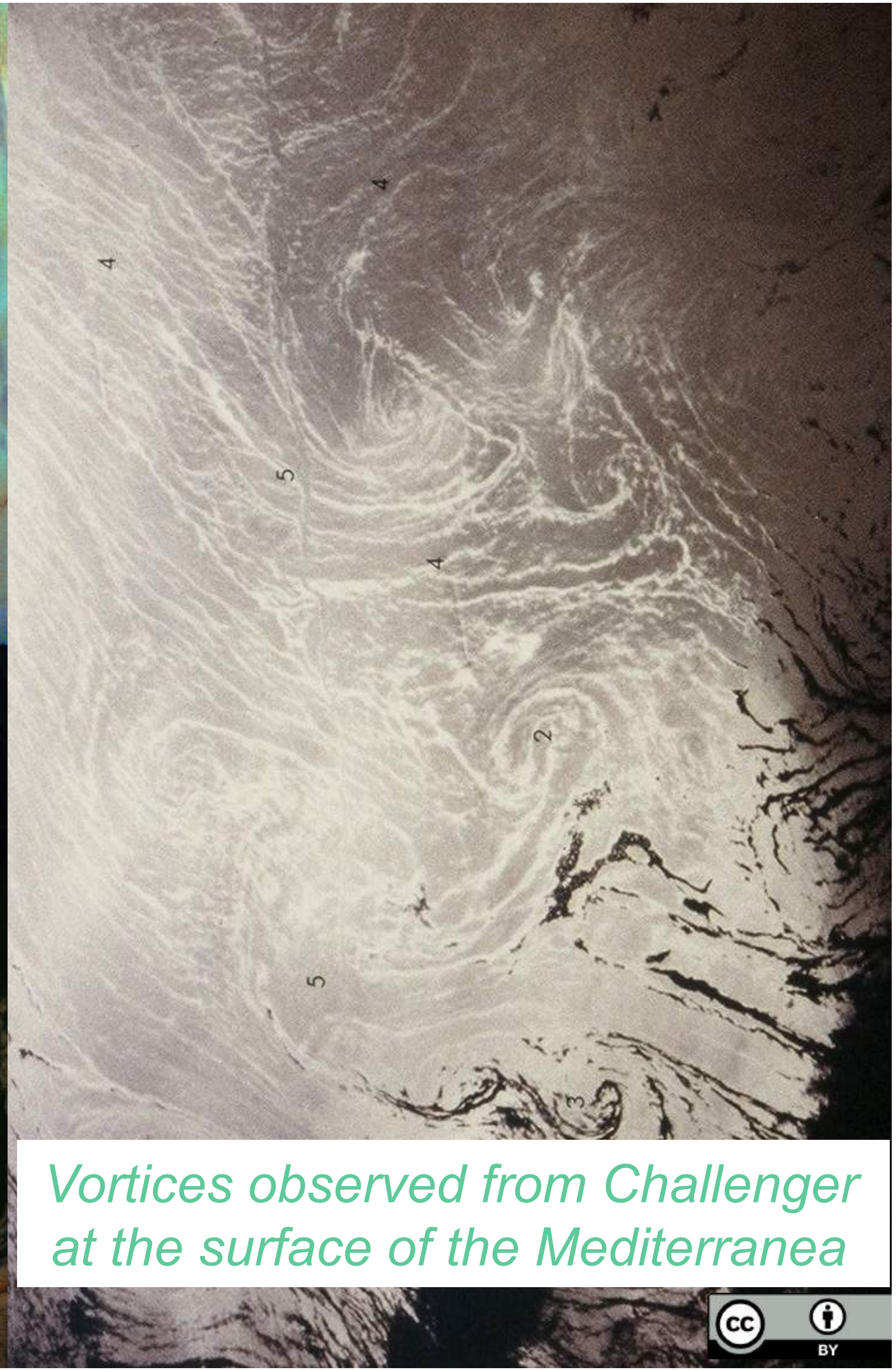




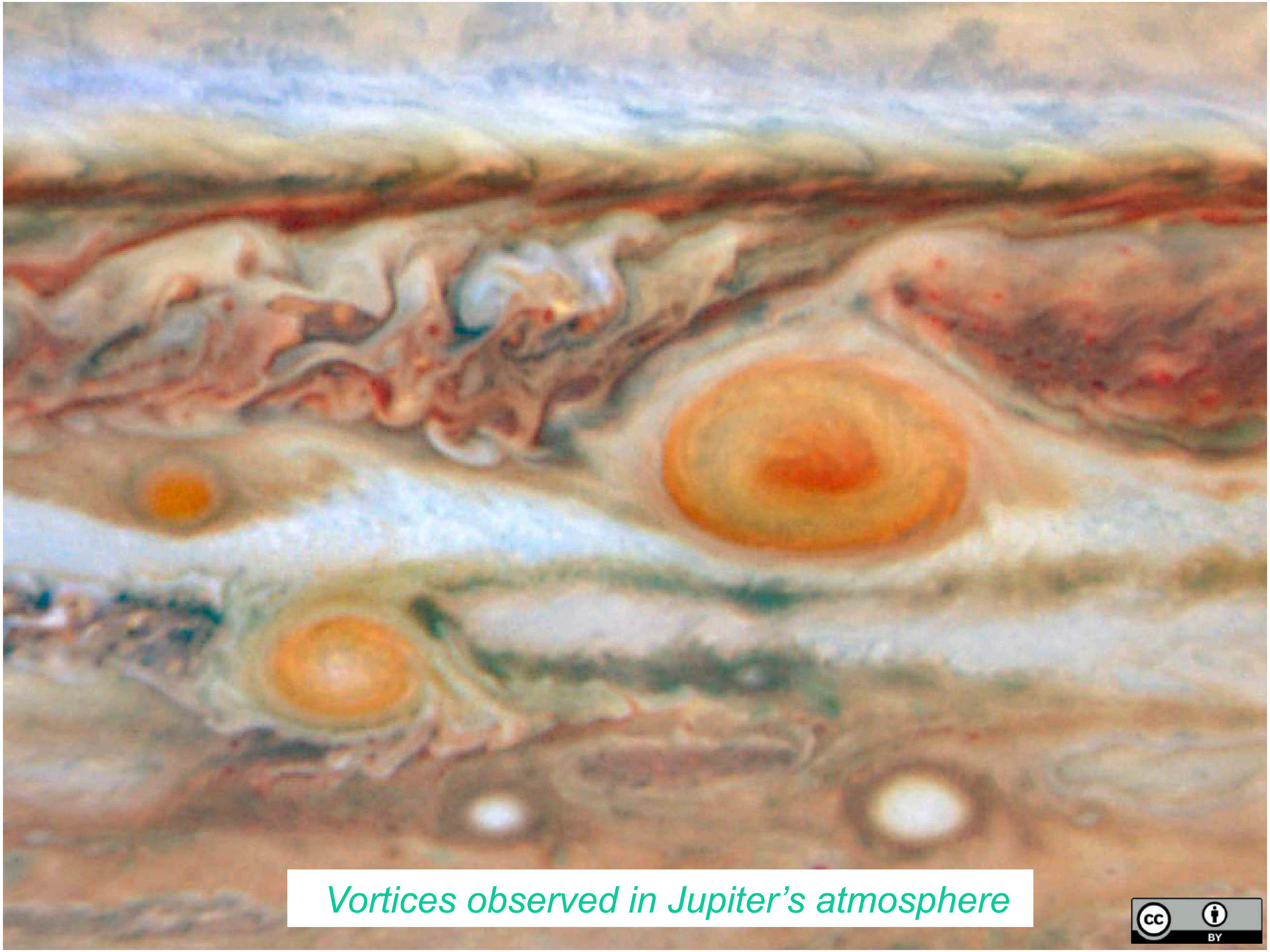
*Clouds tracing the air flow
and plancton tracing the water flow
on the coast of California*



Clouds tracing the vortices emitted In the wake of Guadalupe Islands

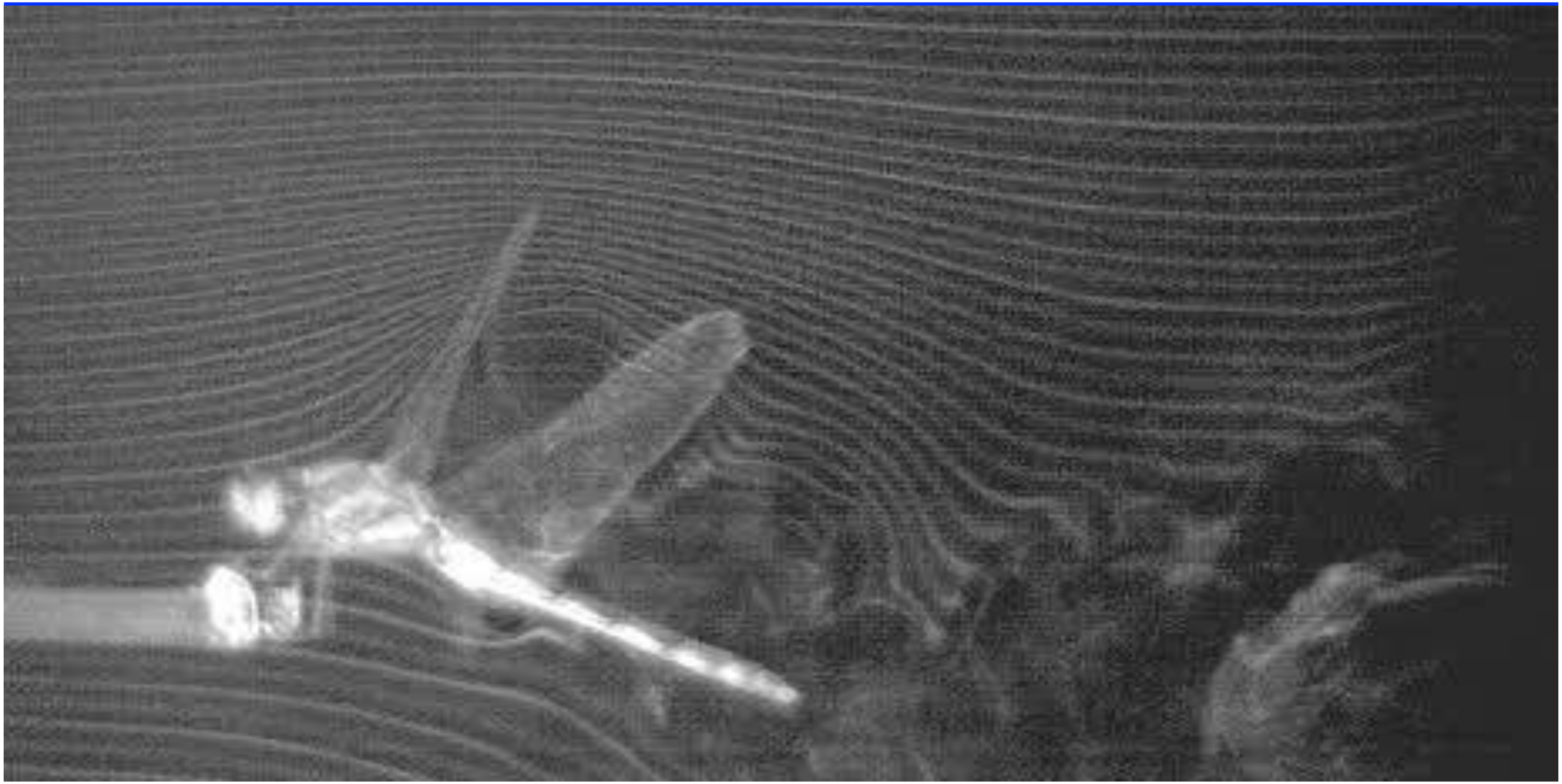


Vortices observed from Challenger at the surface of the Mediterranea



Vortices observed in Jupiter's atmosphere

Observe - Question - Represent



Smoke tracing the air flow to visualize the vortices emitted by a dragon fly

Fundamental principles and equations

Mathematicians and `natural philosophers', such as *d'Alembert* (1752), *Euler* (1758), *Navier* (1822), *Stokes* (1845), designed the fundamental equations of fluid mechanics from the conservation of the flow momentum and the hypothesis of the fluid incompressibility.

$$\Longrightarrow$$

Navier-Stokes equations :

$$\partial_t \vec{\omega} + (\vec{v} \cdot \nabla) \vec{\omega} - \vec{\omega} \cdot \nabla \vec{v} = \nu \nabla^2 \omega + \nabla \times \vec{F}$$

with $\vec{\omega} = \nabla \times \vec{v}$ and $\nabla \cdot \vec{v} = 0$

Flow variables: v velocity, ω vorticity, F external forces.

Fluid parameters: ν kinematic viscosity and density ρ=1.

Plus initial and boundary conditions.

Turbulence: limit where nonlinear transport dominates linear dissipation.

Euler equations: fluid is inviscid $\nu = 0$, therefore no energy dissipation.

How to solve Navier-Stokes equations ?

For the turbulent regime we do not even know
if their solutions exist and are unique for all times

$$\Longrightarrow$$

In the absence of anything better to do,
we try to break ground by looking for approximate solutions,
which we explore by numerical experiment.

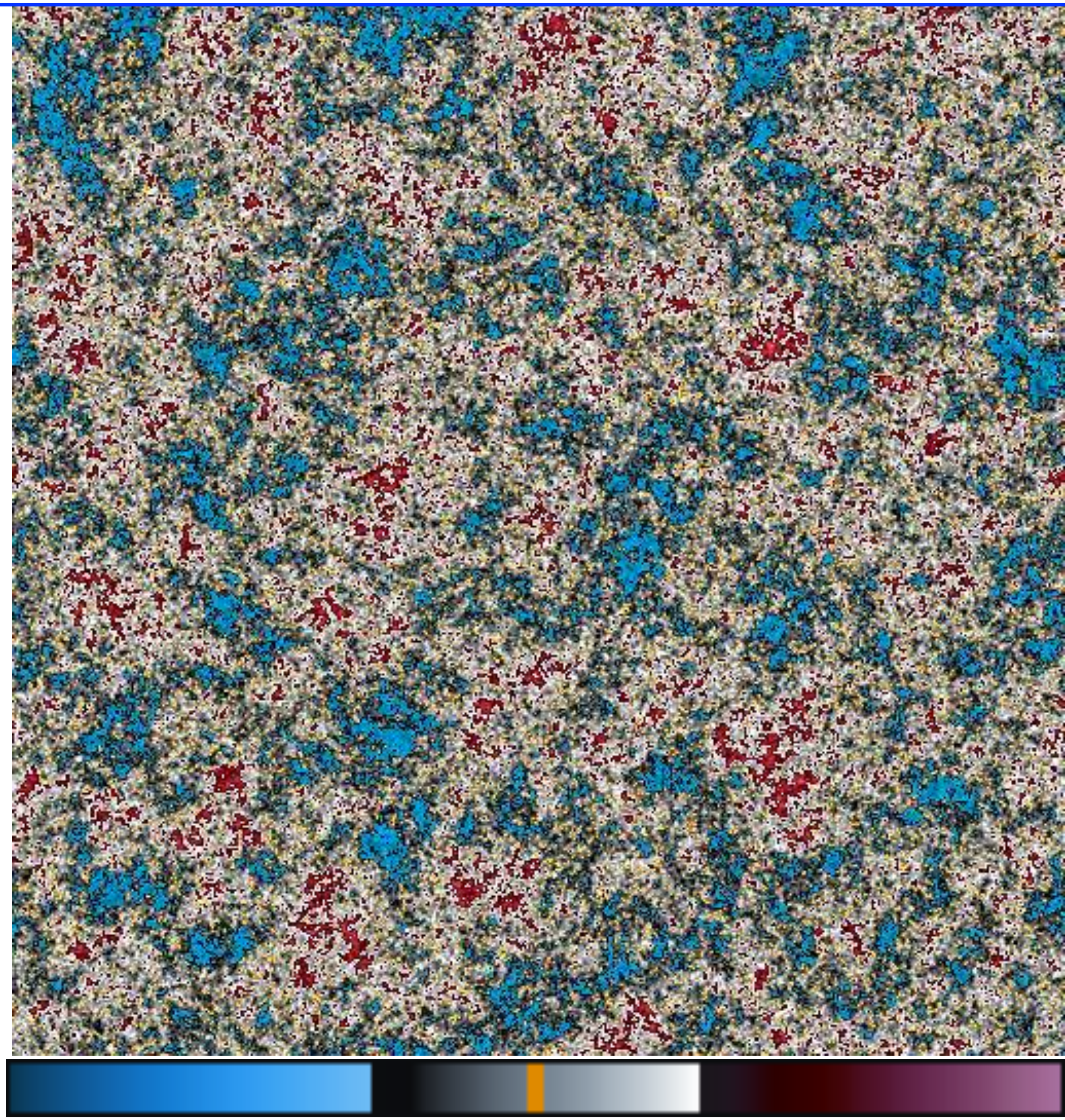
Periodic 2D turbulent flow without forcing

Resolution
$N=512^2$

Gaussian random initial condition

Pseudo-spectral method with a 2/3 dealiasing

M. F., 1988
Fluid Dynamics Research, ***3***



ω *min* $\qquad$ ω *max*

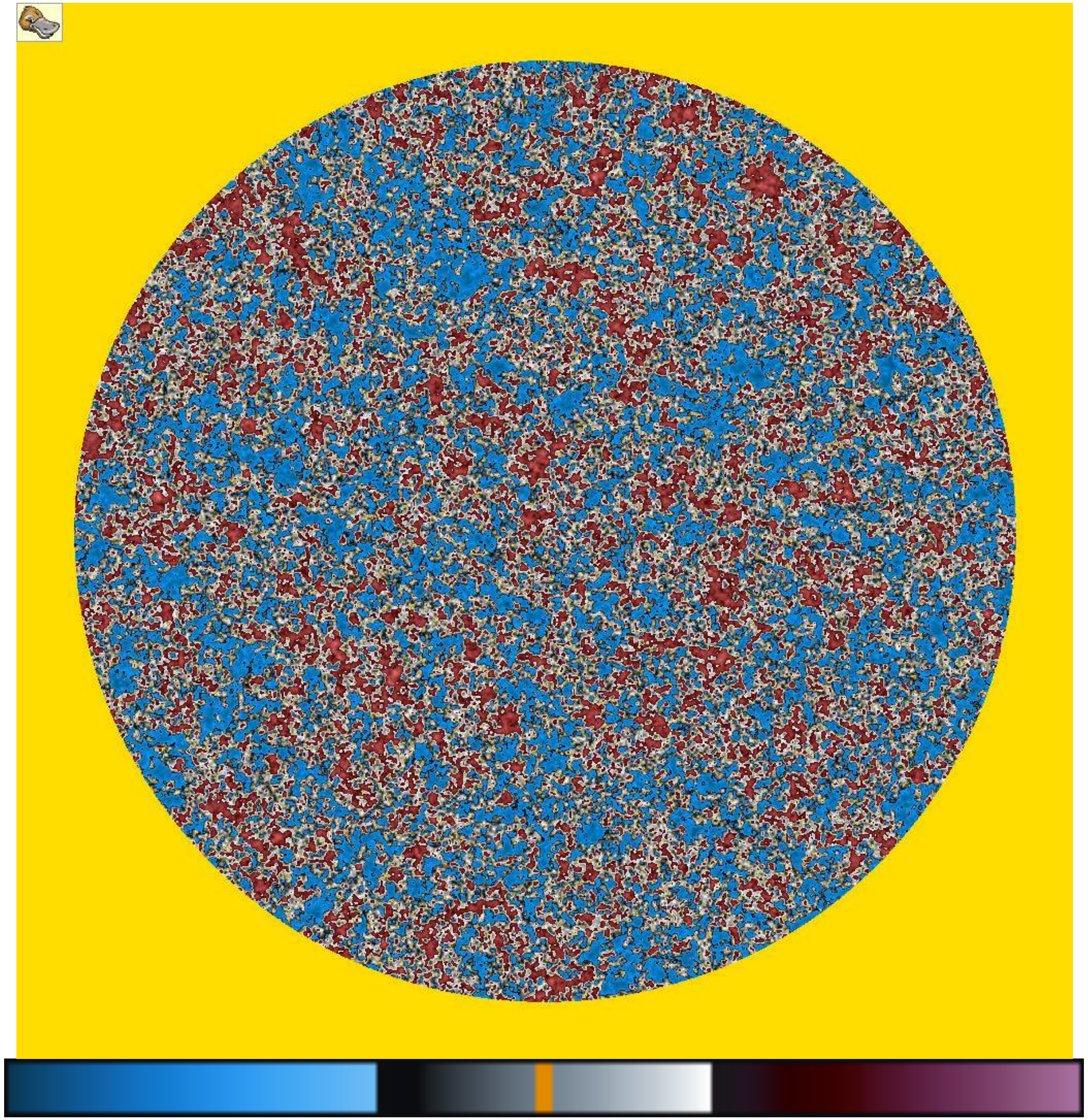
Time evolution of the vorticity field from random Initial conditions, without external forcing

Confined 2D turbulent flow without forcing

Résolution
N=1024^2

Gaussian random initial condition

Pseudo-spectral method with volume penalization



Kai Schneider and M. F., 2005 Phys. Rev. Lett., 95

What is turbulence ?

Turbulence is a state that afluid flow reaches
when it becomes unstable and highly fluctuating.

Etymology of the word 'turbulence'

turba-ae : crowd, mob,
turbo-inis : vortex.

$\Longrightarrow$

A turbulent flow is a mob of vortices interacting together on a wide range of temporal and spatial scales.

Hypotheses :

- we suppose the fluid to be a continuous medium if the scale of the observer it is much larger than the mean free path of molecules,
- Here, we consider the fluid to be incompressible, *i.e.*, non-divergent.

Fluid flows reach the fully-developed regime
when they become highly mixing,
which corresponds to strong turbulence.

The different flow regimes

Deterministic predictability

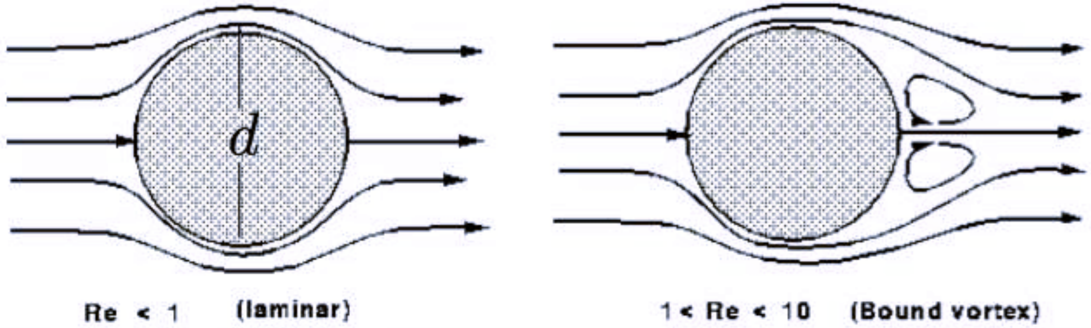
Steady flow

Re ~ 1 to 10^2

$$\mathrm{Re} = \frac{u\,d}{\nu}$$

Laminar

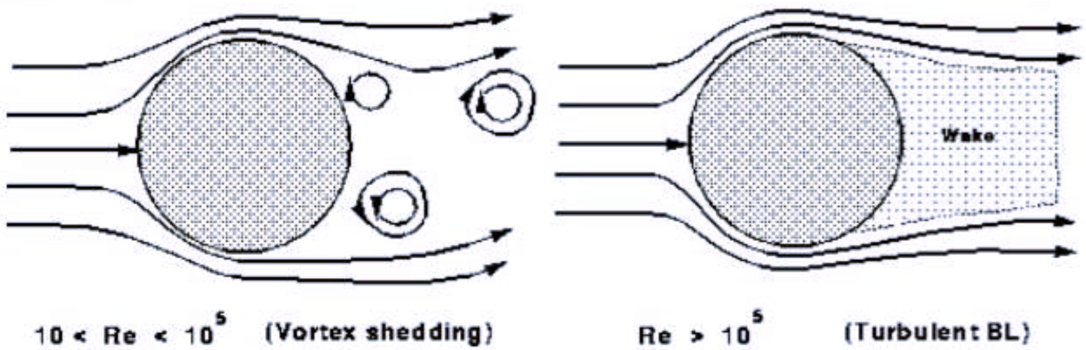
inflow u



Transition

Weakly turbulent

inflow u



Strongly turbulent and 'mixing'

for Re > 10^5

Statistical predictability

Unsteady flow

Re ~ 10^2 to 10^{10}

G. I. Taylor, 1938

'The fact that small quantities of very high frequency disturbances appear, and increase as the speed increases, seems to confirm the view frequently put forward by the author that the dissipation of energy is due chiefly to the formation of very small regions where the vorticity is very high.'

Taylor 1938,
'The spectrum of turbulence',
Proc. Royal Soc. London A, ***164***

Turbulent flows are intermittent,
i.e., the sparser their fluctuations the stronger they are.

$\Longrightarrow$

We should focus on the vorticity field and study
how intermittent structures, such as vortices, emerge and interact.

The Fourier representation is adequate to analyze waves, but not vortices.

Vortices should be analyzed in physical space and in wavelet space.

WHAT IS THE WAVELET REPRESENTATION ?

An adequate representation for music

'If we consider a musical piece and that a note, for instance an A, appears at least once in it, the harmonic analysis will represent the corresponding frequency with a certain amplitude and a certain phase, but without localizing the A in time. However, it is obvious that during this musical piece there are instant for which we do not hear the A note. Although the Fourier representation is mathematically correct, because the phases of nearby notes are organized in such a way that they destroy by interference the A when we do not hear it or reinforce it, also by interference, when we hear it, but, if there is in this conception a skill which honors mathematical analysis, we should not hide the fact that there is a deformation of reality: indeed, when we do not hear the A, the genuine reason is that the A note has not been emitted.'

Jean Ville,
Théorie et application de la notion de signal analytique,
Cables et transmissions, 1948

Continuous Fourier transform

$f(x) \in L^1(R) \cap L^2(R)$ is integrable and square integrable

Analysis

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi k \cdot x} dx$$

Synthesis

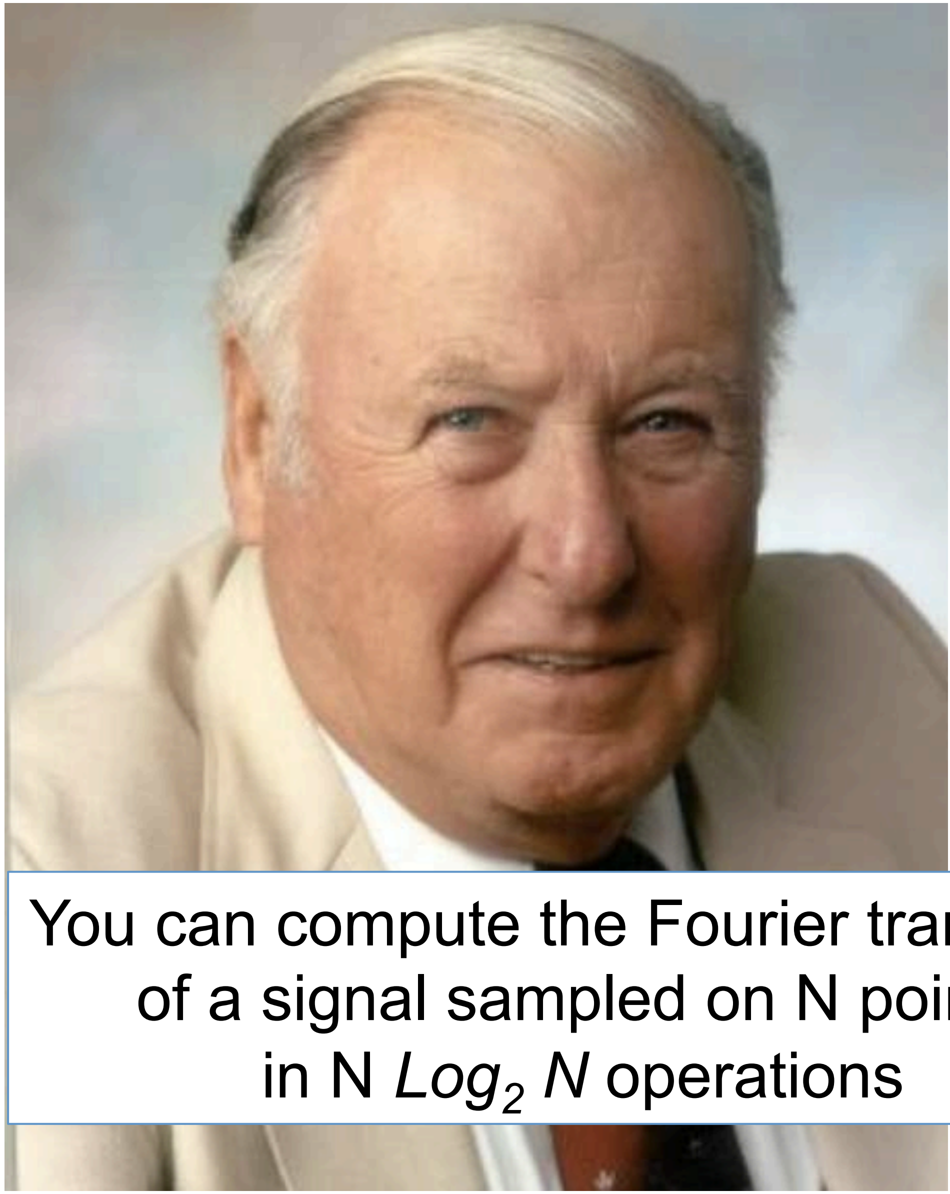
$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(k) e^{i2\pi k \cdot x} dk$$

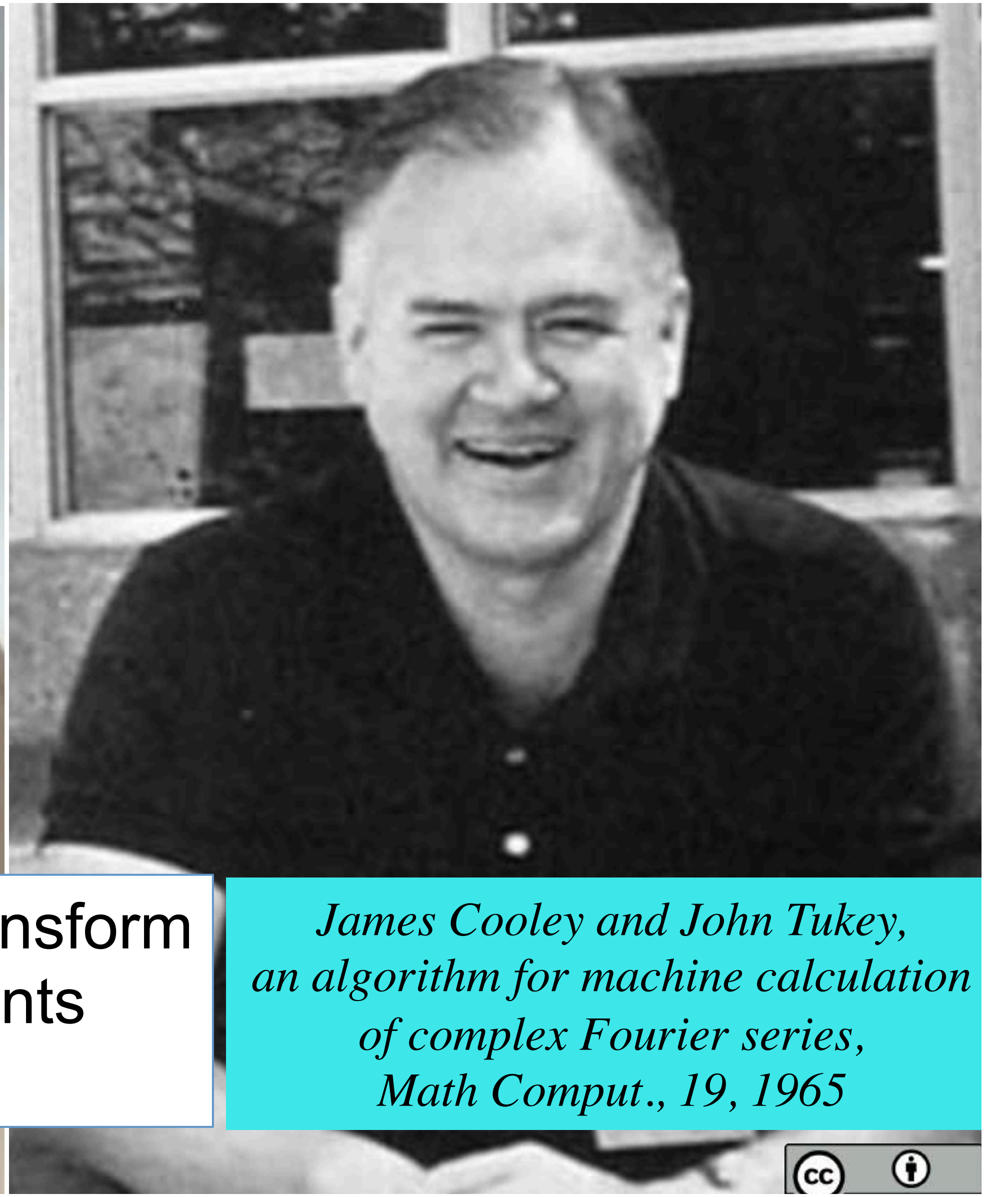
Parseval's identity (energy conservation)

$$\int_{-\infty}^{\infty} f_1(x) \cdot f_2(x) dx = \int_{-\infty}^{\infty} \widehat{f_1}(k) \cdot \widehat{f_2}(-k) dk$$

James William Cooley
(1926-2016)

John Tukey
(1915-2000)





You can compute the Fourier transform of a signal sampled on N points in N $Log_2 N$ operations

James Cooley and John Tukey, an algorithm for machine calculation of complex Fourier series, Math Comput., 19, 1965

Continuous wavelet transform

Analysis

$$\widetilde{f}(a,b) = \int_{-\infty}^{\infty} f(x)\psi^{*}_{a,b}(x)\,dx$$

Synthesis

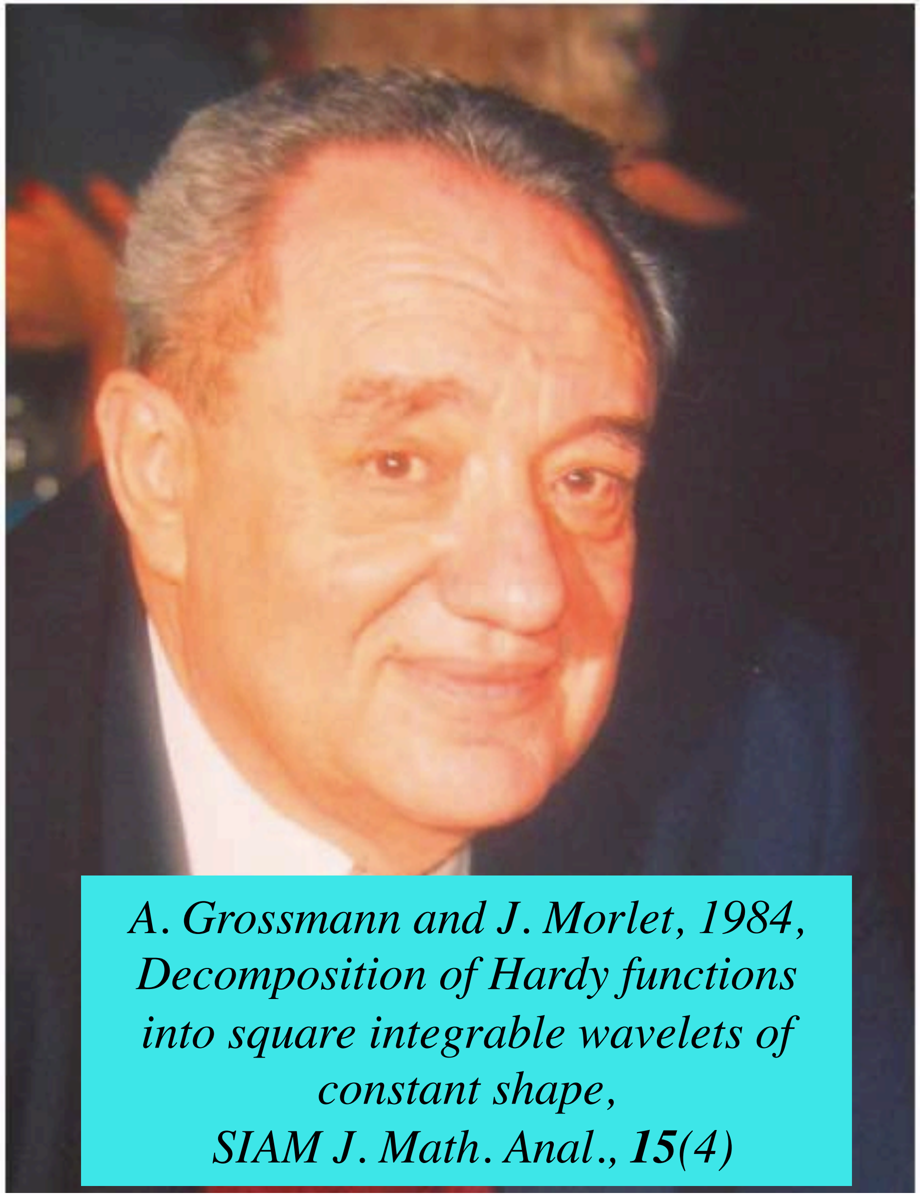
$$f(x) = \frac{1}{C_\psi} \int_0^{\infty} \int_{-\infty}^{\infty} \widetilde{f}(a,b)\,\psi_{a,b}(x)\,\frac{da\,db}{a^2}$$

Parseval's identity (energy conservation)

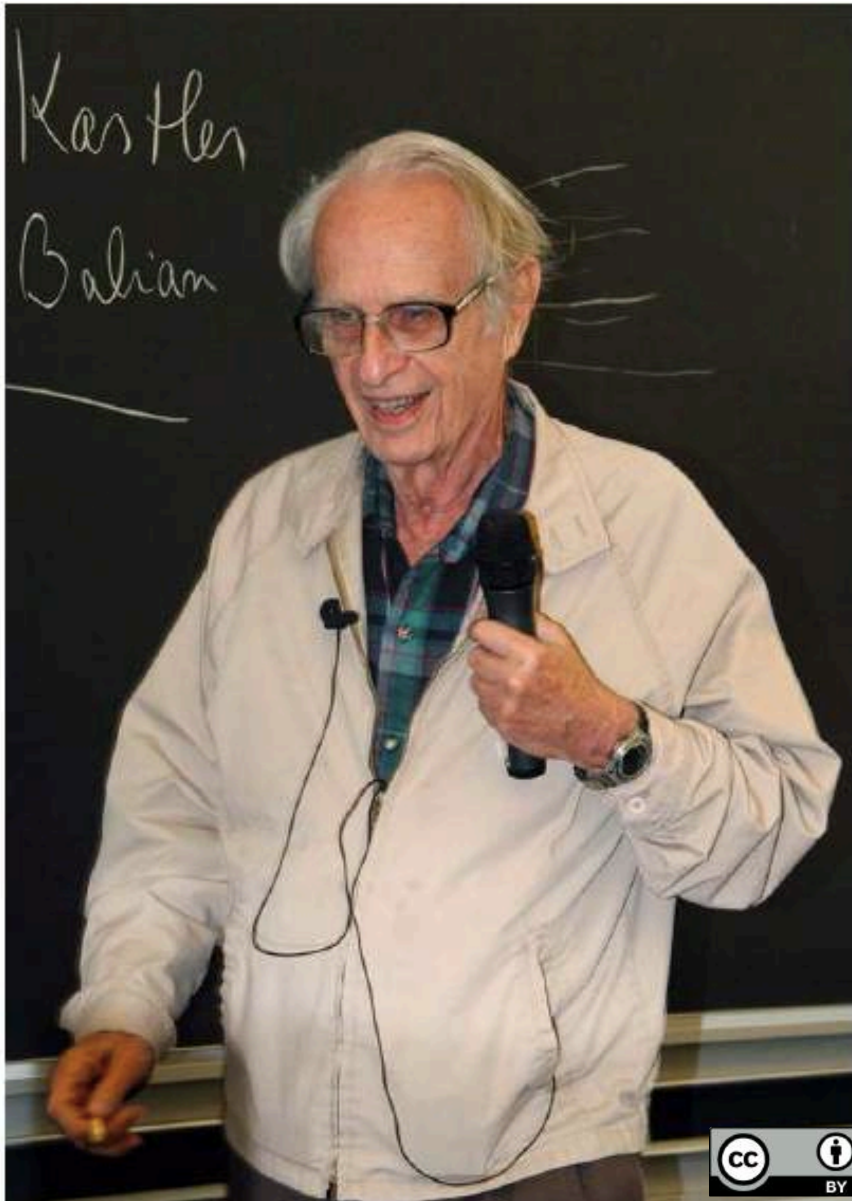
$$\langle f_1, f_2 \rangle = \int_{-\infty}^{\infty} f_1(x)\, f_2^{*}(x)\,dx = \frac{1}{C_\psi} \int_0^{\infty} \int_{-\infty}^{\infty} \widetilde{f}_1(a,b)\,\widetilde{f}_2^{\,*}(a,b)\,\frac{da db}{a^2}$$

Jean Morlet
(1931-2007)

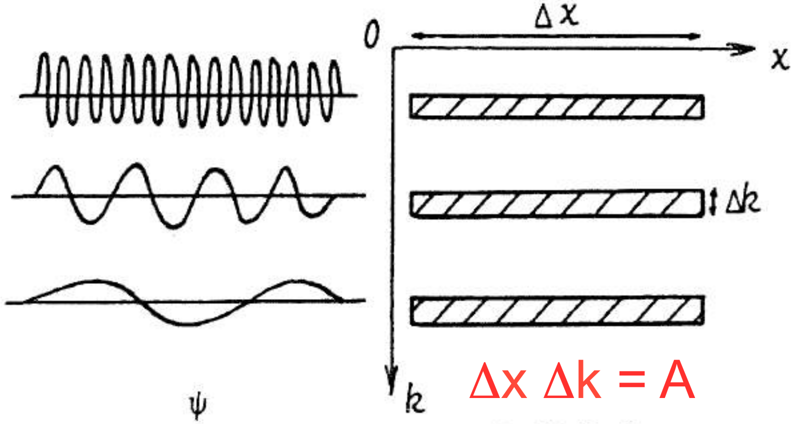
Alex Grossmann
(1930-2019)



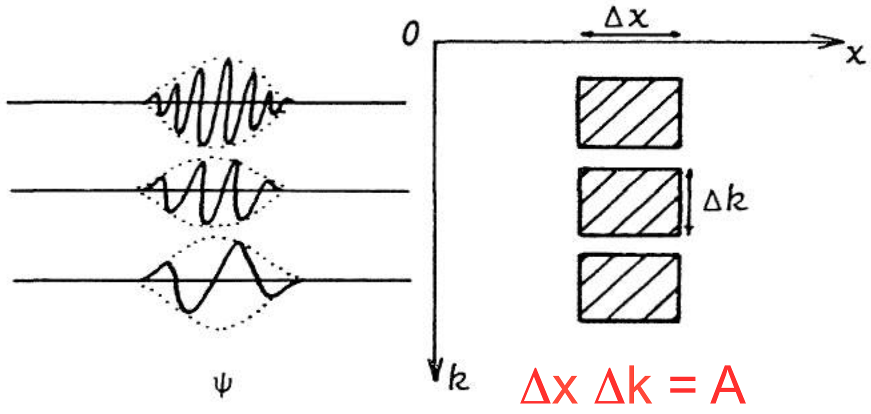
A. Grossmann and J. Morlet, 1984,
Decomposition of Hardy functions
into square integrable wavelets of
constant shape,
SIAM J. Math. Anal., ***15****(4)*



Fourier versus wavelet transform



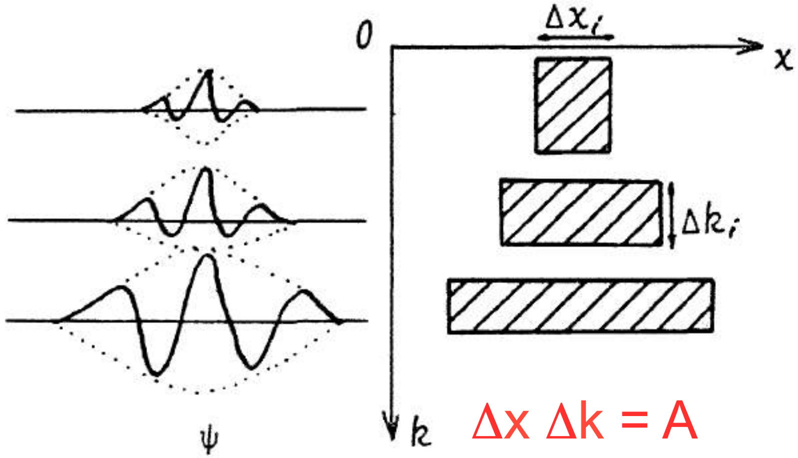
Modulation
Fourier, 1822



Translation and modulation
Windowed Fourier, Gabor, 1946

Wavelet representation
is optimal regarding
the uncertainty principle
$\Delta x \, \Delta k = A > 0$

M. F., C. R. Acad. Sci.
Paris, 1988

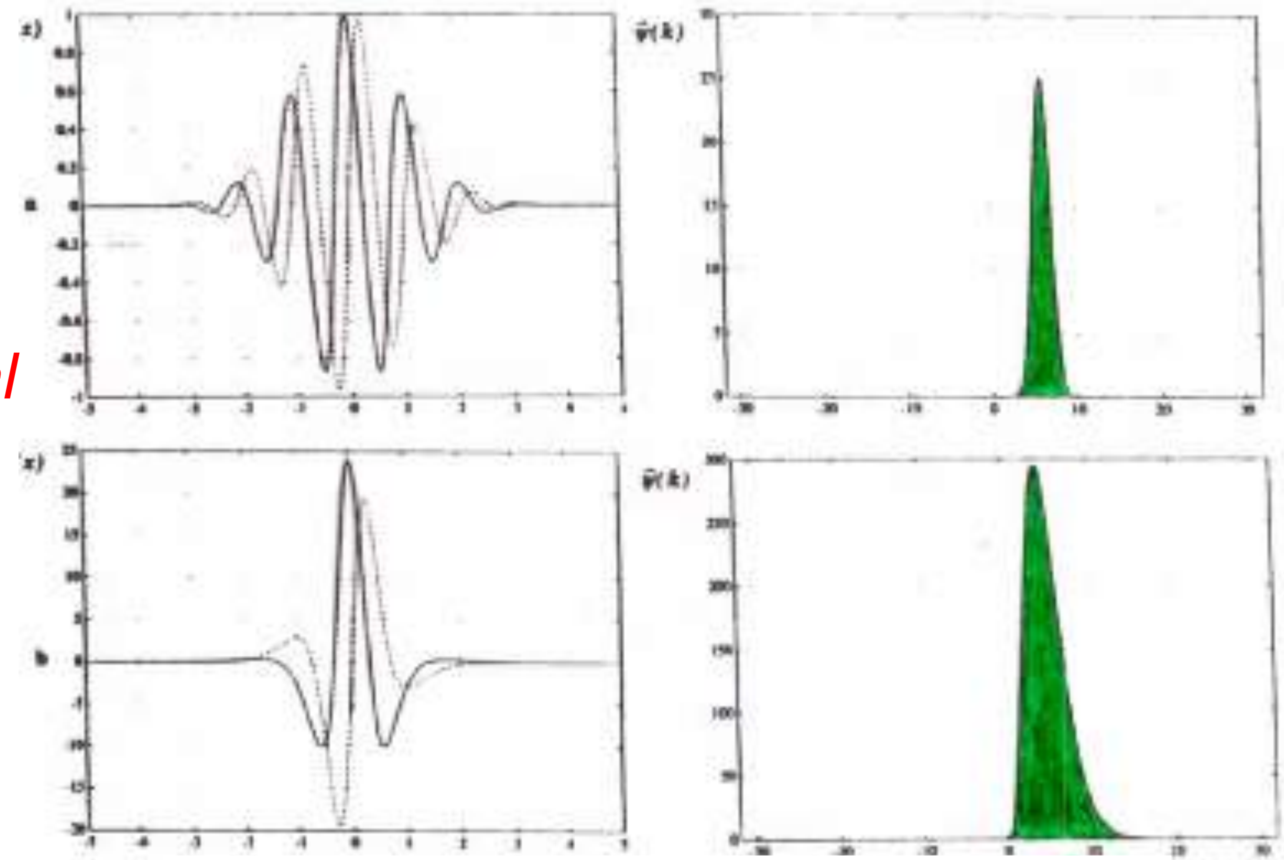


Translation and dilatation
Morlet, 1981

Complex-valued wavelets

Morlet's wavelet

Wavelet in physical space

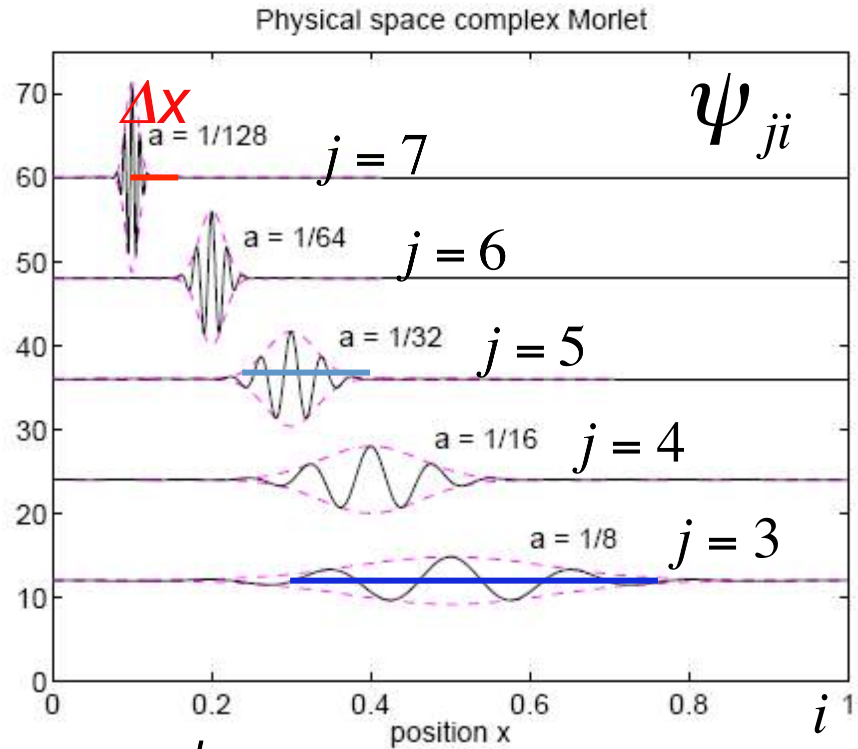


Wavelet in spectral space

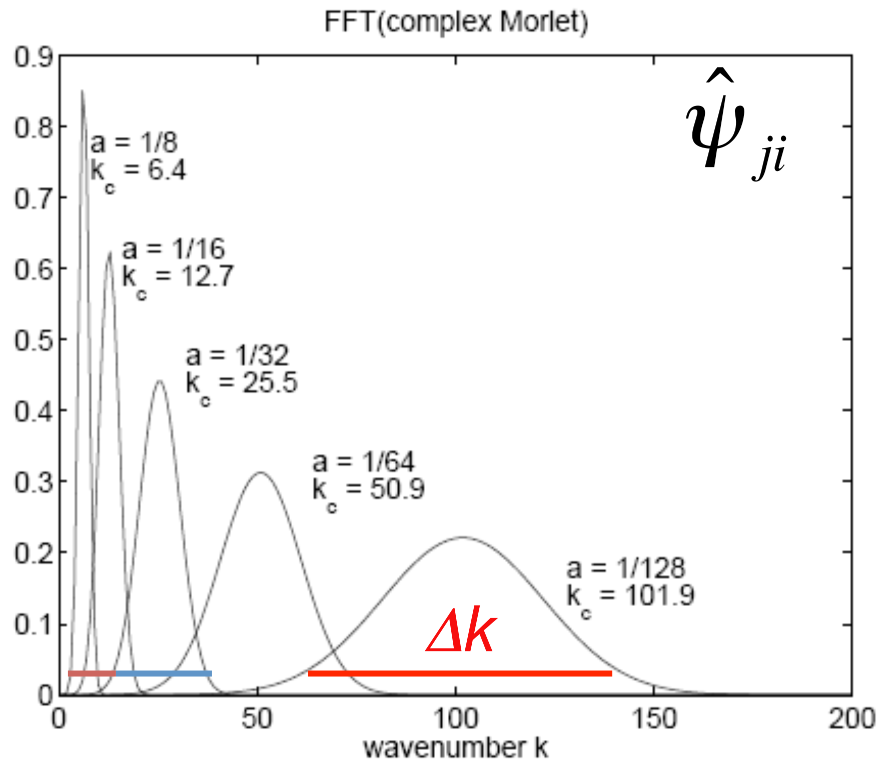
Paul's wavelet

Generation of the wavelet 'family'

Small scales

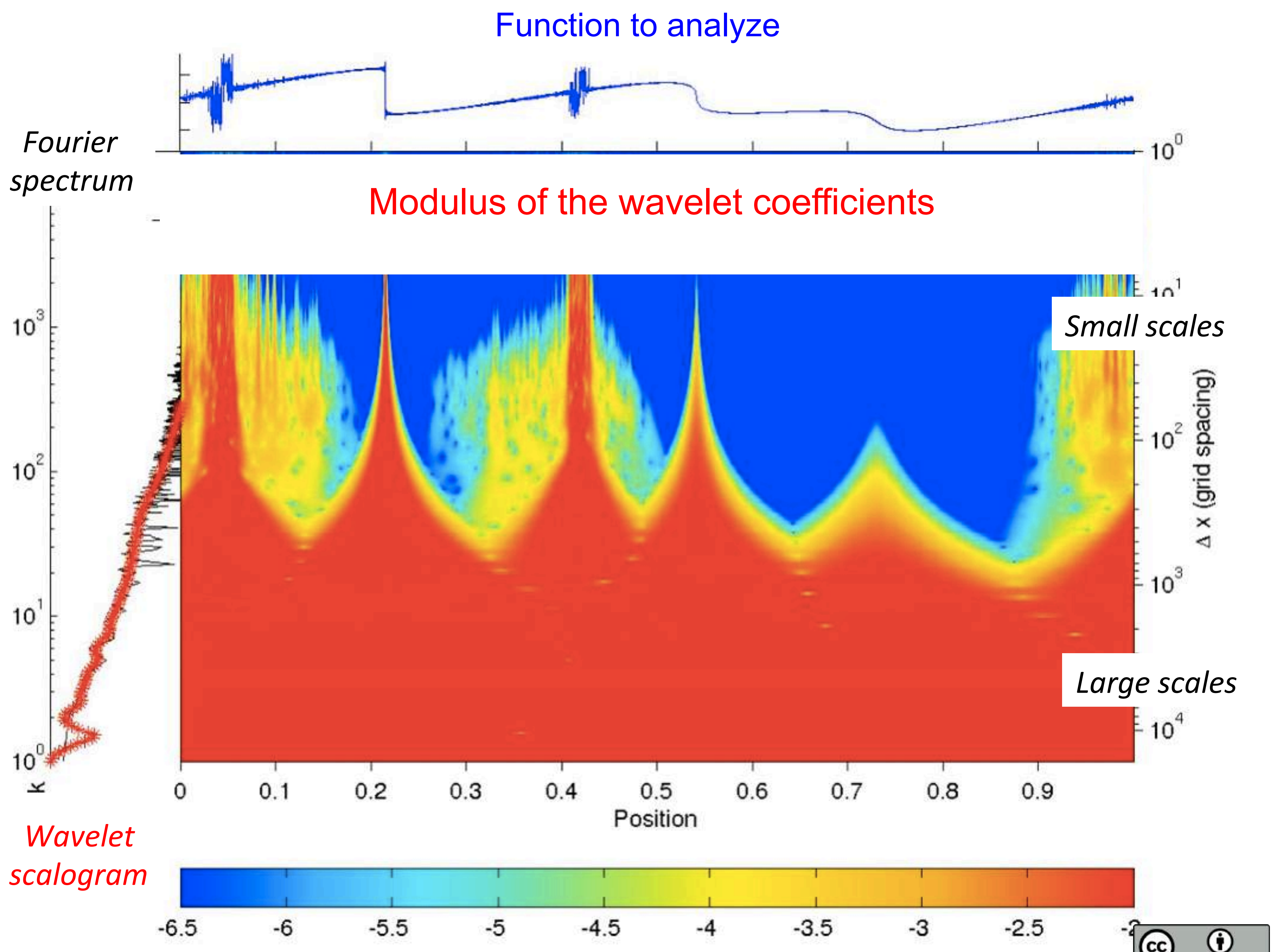


Large scales



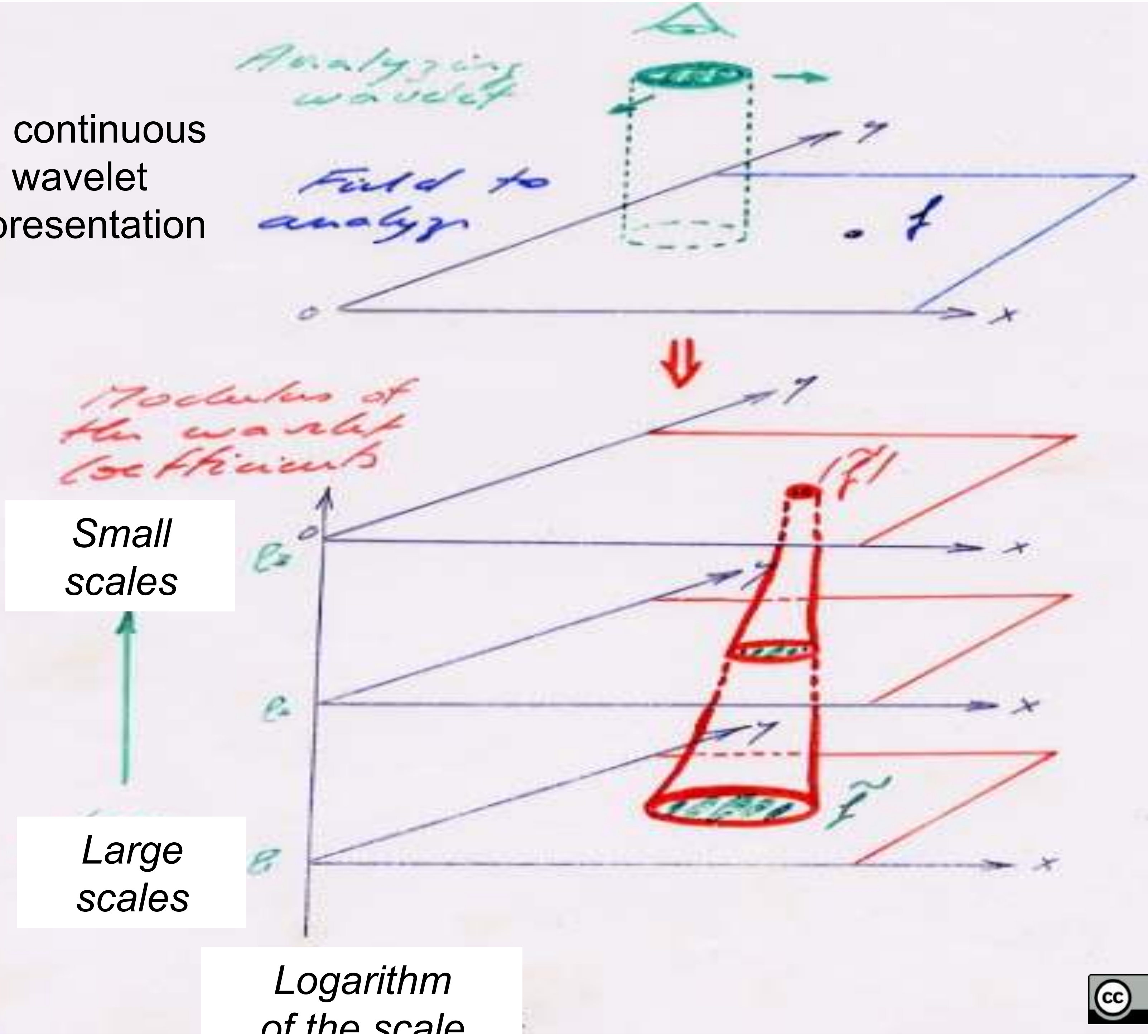
Physical space *Spectral space*

$$\Delta x \, \Delta k > A$$



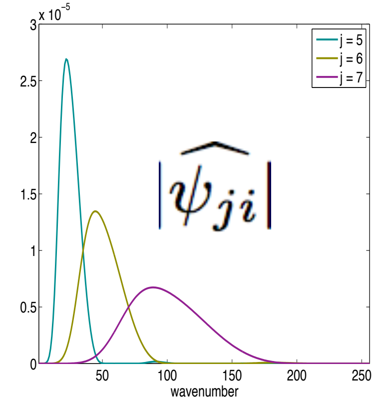
Function to analyze
Fourier spectrum
Modulus of the wavelet coefficients
Small scales
Large scales
Δ x (grid spacing)
k
Position
Wavelet scalogram
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
-6.5 -6 -5.5 -5 -4.5 -4 -3.5 -3 -2.5 -2

2D continuous wavelet representation



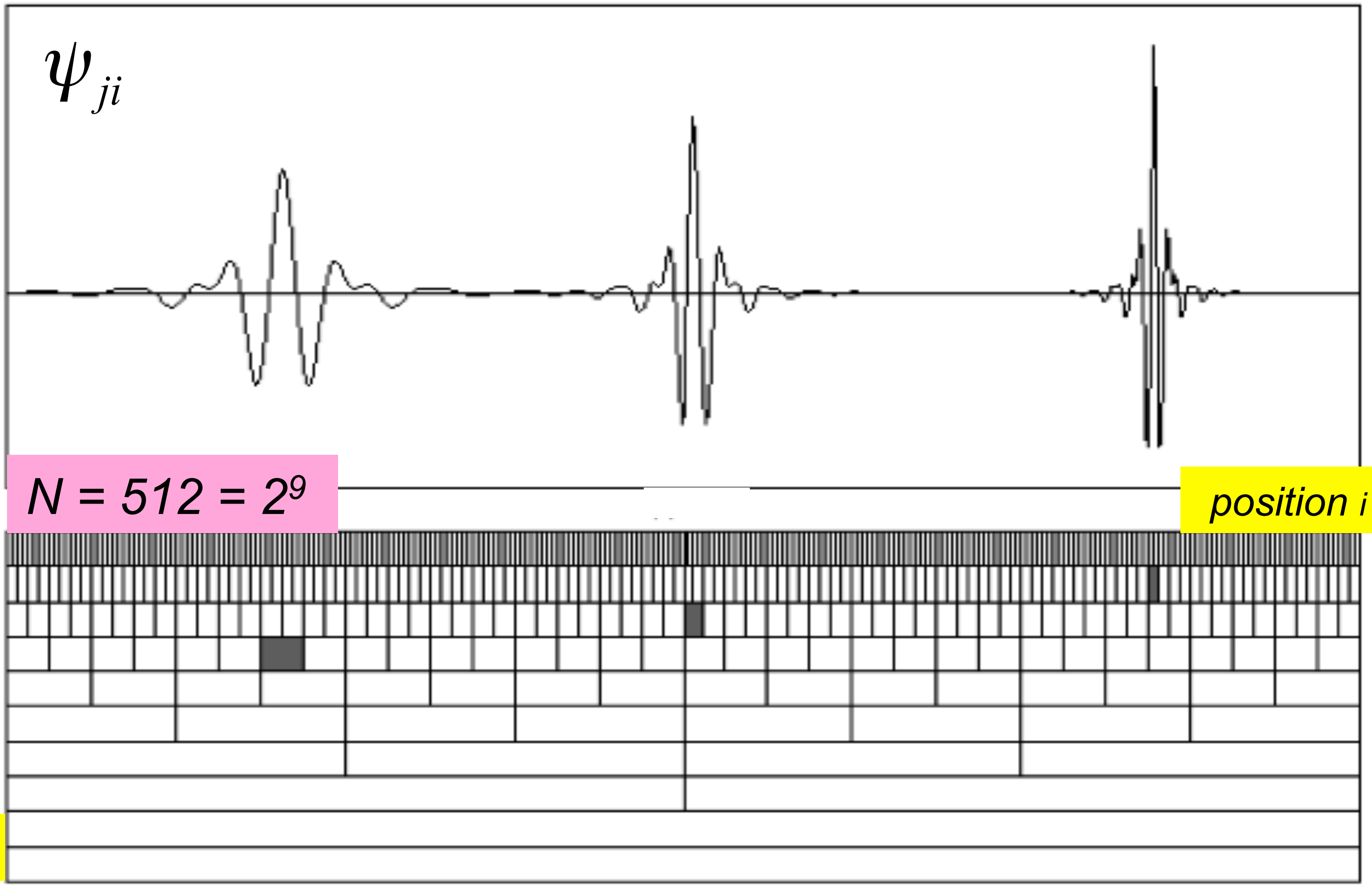
Orthogonal wavelet representation

Wavelets



wavelet coefficients

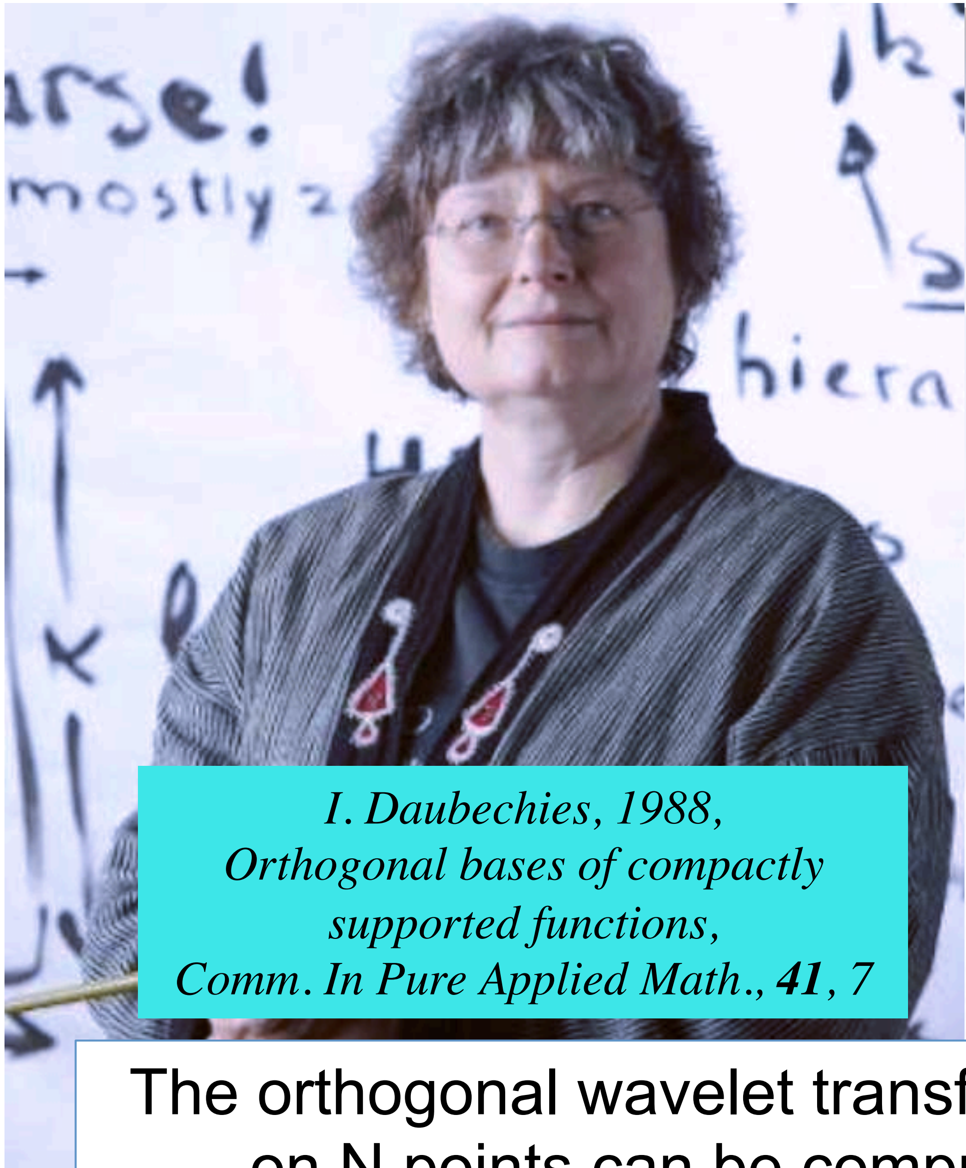
$$\tilde{f}_{ji} = \langle \psi_{ji} | f \rangle$$



log of scale j

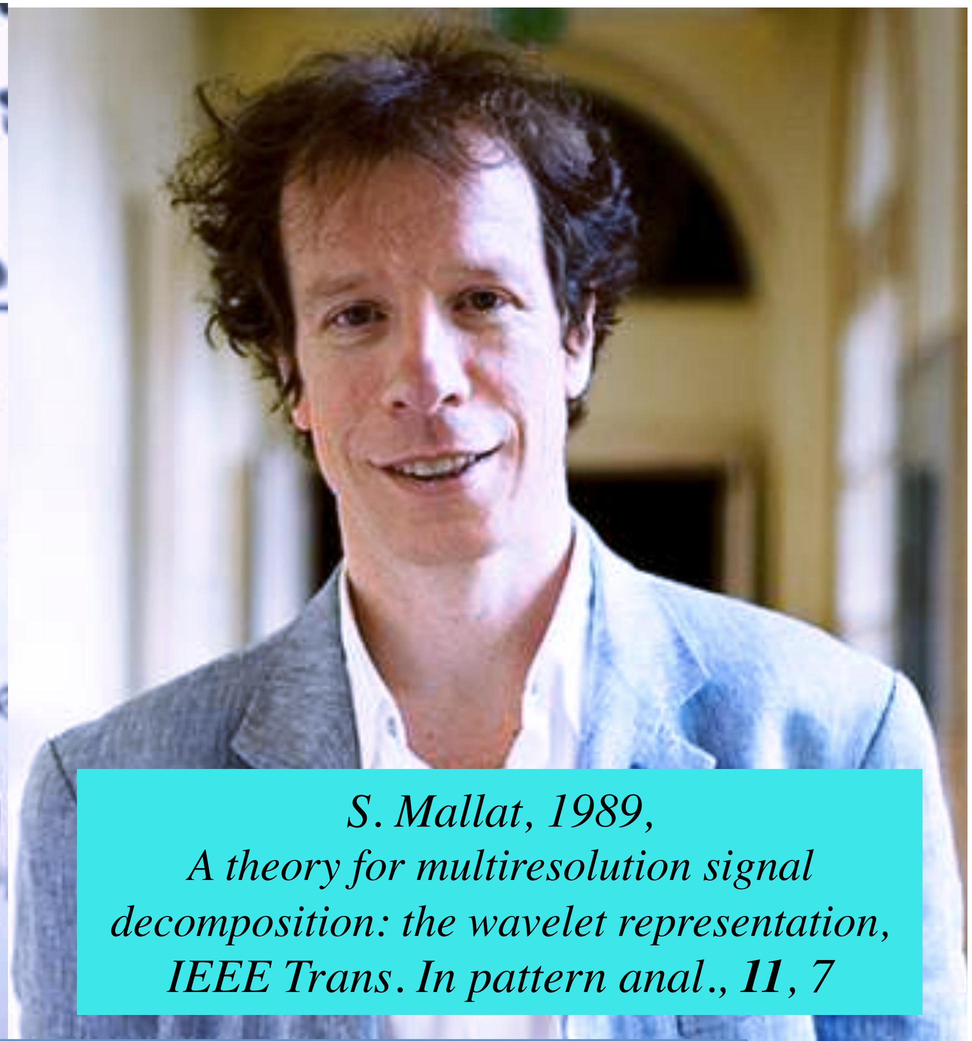
Mallat, 2008
A wavelet tour of signal processing
Academic Press

Ingrid Daubechies



I. Daubechies, 1988,
Orthogonal bases of compactly supported functions,
Comm. In Pure Applied Math., ***41****, 7*

Stéphane Mallat

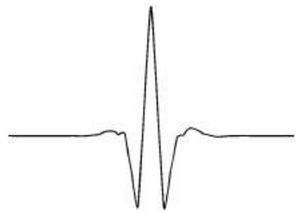


S. Mallat, 1989,
A theory for multiresolution signal decomposition: the wavelet representation,
IEEE Trans. In pattern anal., ***11****, 7*

The orthogonal wavelet transform of a signal sampled on N points can be computed in N operations

Continuous / orthogonal wavelets

Analyzing functions are
translates and dilates
of an oscillating function (of zero mean)



Well localized in both space and wavenumber

$$\tilde{f}(l, \vec{x}) = \langle \psi_{l,\vec{x}} | f \rangle$$

Continuous wavelets	Orthogonal wavelets
$\psi_{l,\vec{x}}(x') = \frac{1}{l^{n/2}} \psi(\frac{\vec{x}' - \vec{x}}{l})$	$\psi_{j,i}(x') = 2^{j/2} \psi(2^j x' - i)$
• Translates and dilates vary continuously • Redundant representation	• Translates and dilates are on a discrete dyadic grid • Orthogonal basis
• Coefficients are easy to read • Unfold in both space and scale • For analysis	• Coefficients not easy to read • sampled on a dyadic grid • For filtering and compression

HOW TO ANALYZE TURBULENT FLOWS ?

How to decompose turbulent flows ?

'In 1938 Tollmien and Prandtl suggested that turbulent fluctuations might consist of two components, a diffusive and a non-diffusive. Their ideas that fluctuations include both random and non random elements are correct, but as yet there is no known procedure for separating them.'

*Dryden, 1948, Adv. Appl. Mech., **1***

mean + turbulent fluctuations
= mean + non random + random
= mean + coherent structures + incoherent noise

⟹ *Coherent Vorticity Extraction (CVE)*

turbulent dynamics
= chaotic non diffusive + stochastic diffusive
= inviscid nonlinear dynamics + turbulent dissipation

⟹ *Coherent Vorticity Simulation (CVS)*

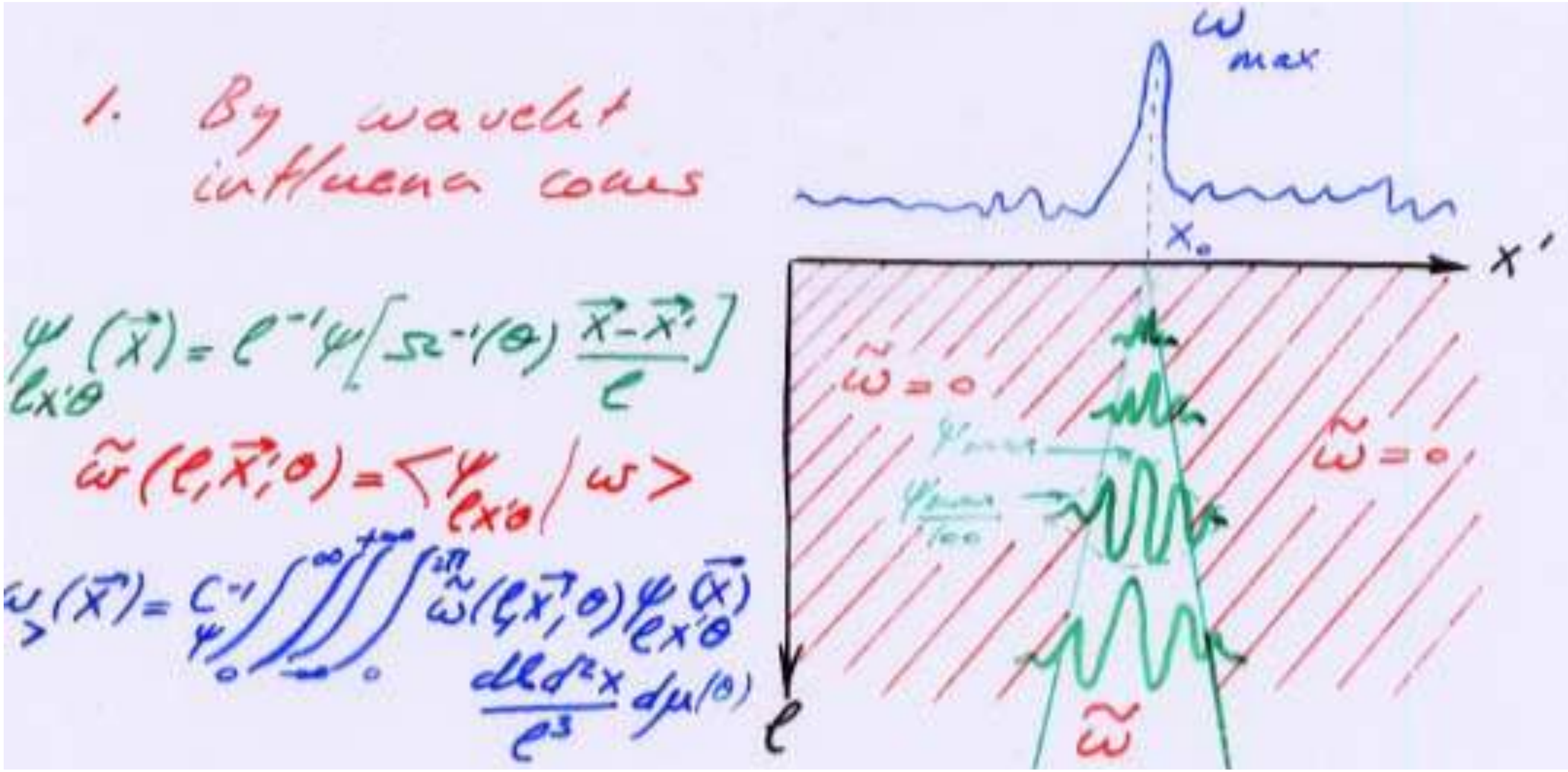
M. F., 1992
*Ann. Rev. Fluid Mech., **24***

M. F., Schneider, Kevlahan, 1999,
*Phys. Fluids, **11** (8)*

M. F., Pellegrino, Schneider, 2001
*Phys. Rev. Lett., **87** (5)*

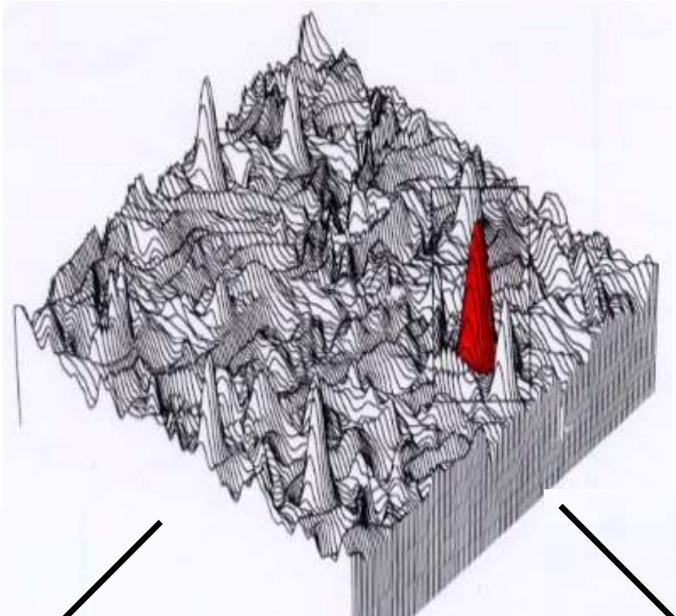
Linear extraction of a coherent structure

To extract a coherent structure which is localized in x_0 we retain all wavelet coefficients in its influence cone, which contains to all wavelets localized in x_0.

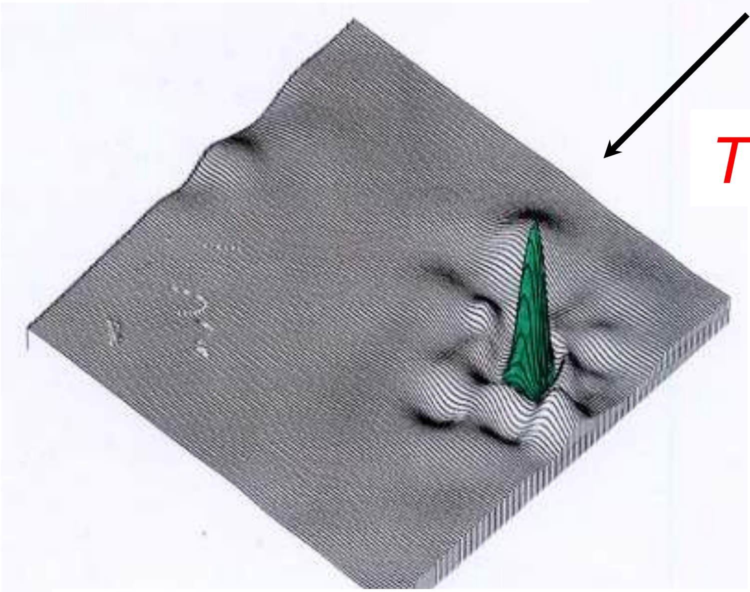


Example of linear extraction in 2D

M. F., 1993
Probability Concepts
in Physical Oceanography,
Hawaii University Press, 131-159

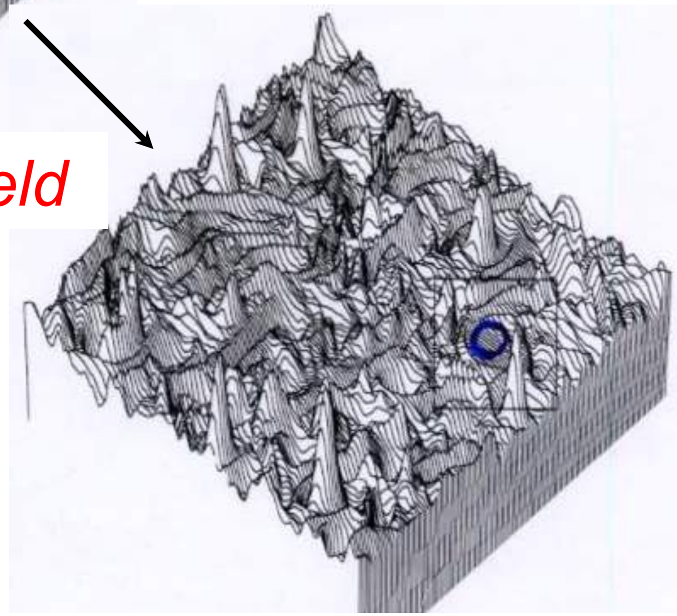


The vorticity field



The field with one vortex

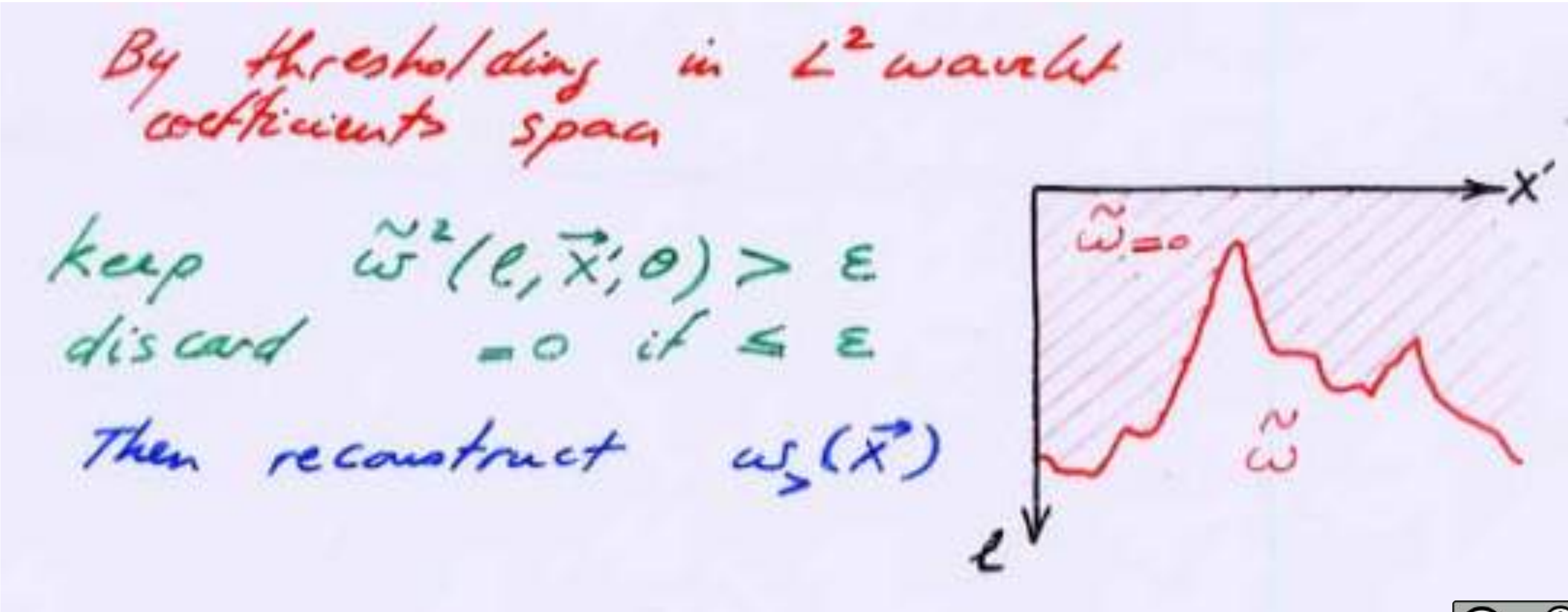
+



The field without one vortex

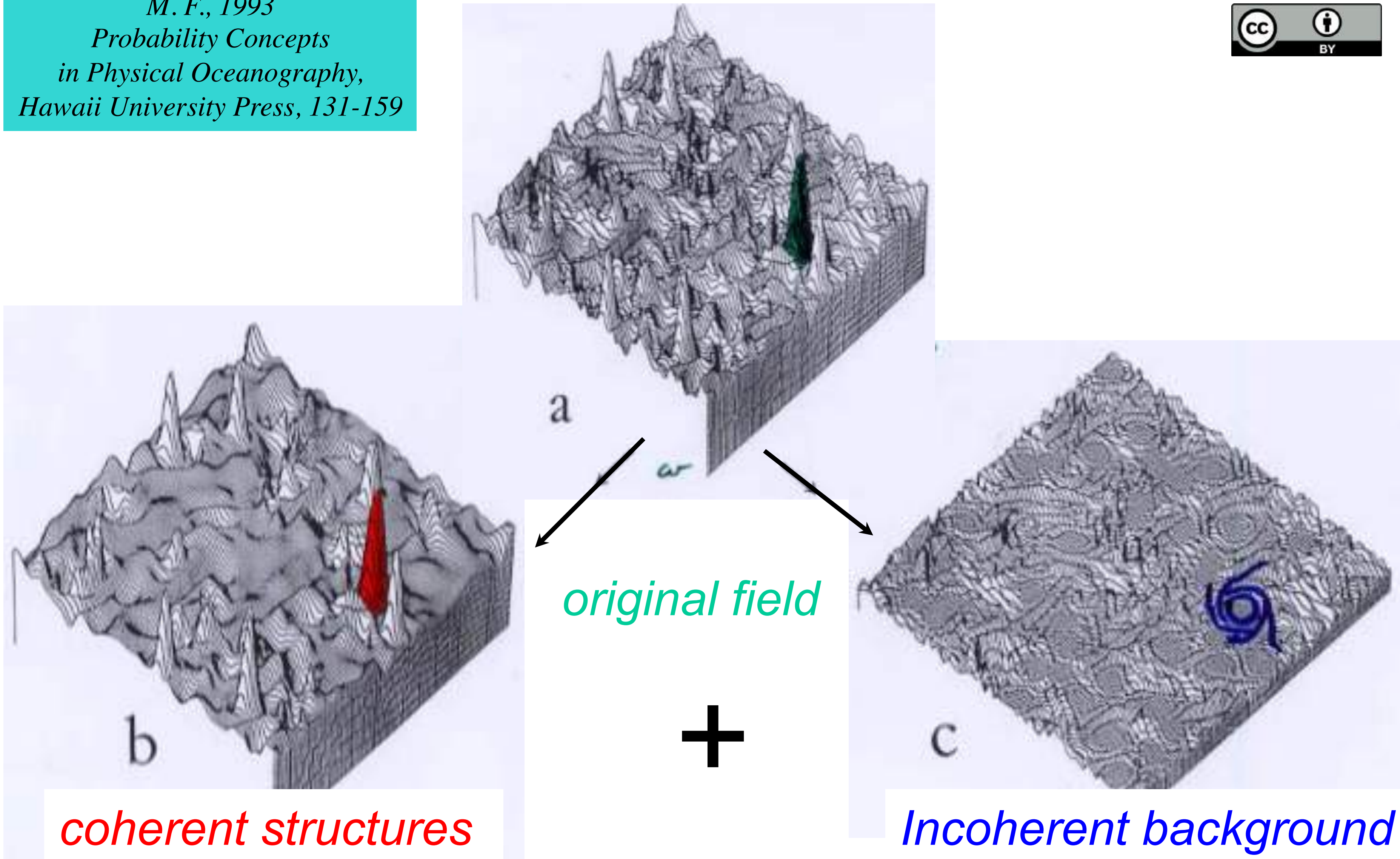
Nonlinear extraction of coherent structures

To extract the most significant stuctures we retain the wavelet coefficients whose modulus is larger than a given threshold value.



Example of nonlinear extraction in 2D

M. F., 1993
Probability Concepts
in Physical Oceanography,
Hawaii University Press, 131-159



A better way to extract coherent structures

Since there is not yet a universal definition of coherent structures
which emerge out of turbulent fluctuations,
we adopt an apophetic method :
instead of defining what they are, we define what they are not.

For this, we propose the minimal statement :
'Coherent structures are not noise'

$\Longrightarrow$

Extracting coherent structures becomes a denoising problem,
not requiring any hypotheses on the structures themselves
but only on the noise to be eliminated.

Choosing the simplest hypothesis as a first guess,
we suppose we want to eliminate an additive Gaussian white noise
and for this we use a nonlinear wavelet filtering.

M.F., Schneider et al., 2003
*Phys. Fluids, **15** (10)*

Azzalini, M. F., Schneider, 2005
*ACHA, **18** (2)*

Wavelet-based denoising algorithm

Apophatic method :

- *no hypothesis on the structures,*
- *only hypothesis on the noise,*
- *simplest hypothesis as our first choice.*

Hypothesis on the noise :

$$f_n = f_d + n$$

n Gaussian white noise,
$\langle f_n^2 \rangle$ variance of the noisy signal,
N number of coefficients of f_n.

Wavelet decomposition :

$$\tilde{f}_{ji} = \langle f \mid \psi_{ji} \rangle$$

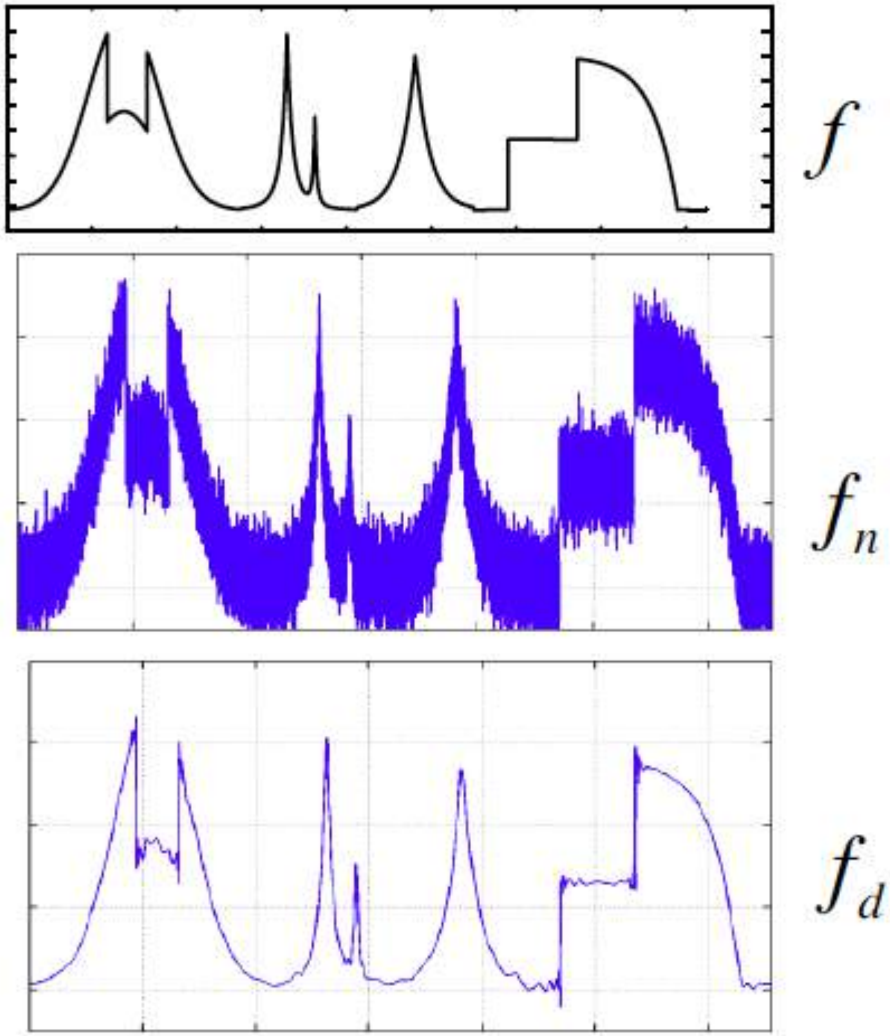
j scale,
i position

Estimation of the threshold :

$$\varepsilon_n = \sqrt{2 \langle f_n^2 \rangle \ \ln(N)}$$

Wavelet reconstruction :

$$f_d = \sum_{ji : |\tilde{f}_{ji}| > \varepsilon_n} \tilde{f}_{ji} \psi_{ji}$$



*Donoho, Johnstone, Biometrika, **81**, 1994*

*Azzalini, M. F., Schneider, ACHA, **18** (2), 2005*

Wavelet filtering of a 2D turbulent flow

PIV
N=128^2

Total vorticity
100% E
100% Z

2% N

Coherent vorticity
99% E
80% Z

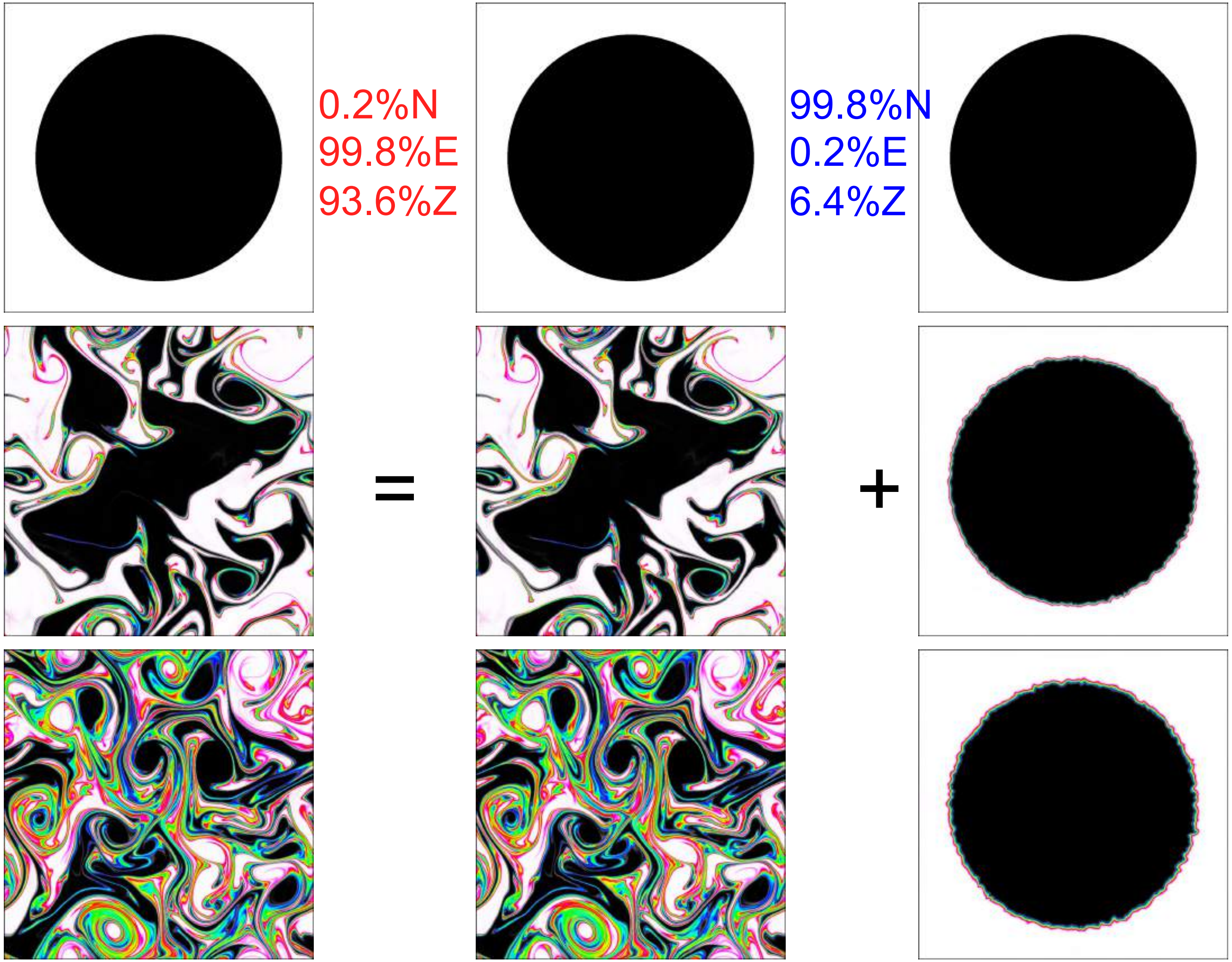
98% N

Incoherent vorticity
1% E
20% Z

Advection of a passive scalar

DNS
$N=512^2$

Transport by the coherent flow and turbulent dissipation by the incoherent flow



Total flow | Coherent flow | Incoherent flow

Beta, Schneider, M.F., 2003, Nonlinear Sci. Num. Simul., 8

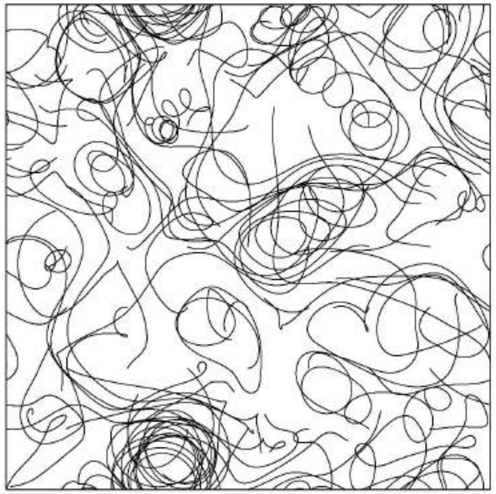
Advection of point particles

DNS
$N=512^2$

0.2 % of coefficients
99.8 % of kinetic energy
93.6 % of enstrophy

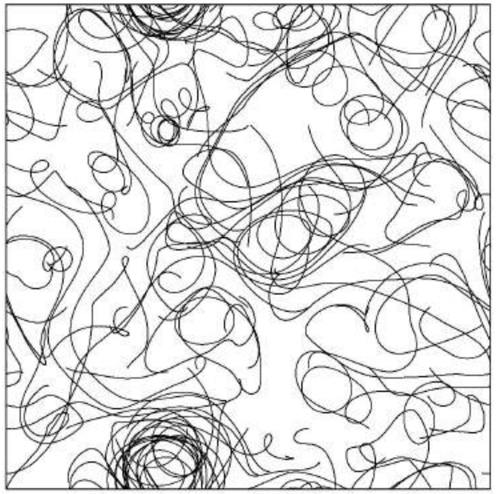
99.8 % of coefficients
0.2 % of kinetic energy
6.4 % of enstrophy

by the total flow



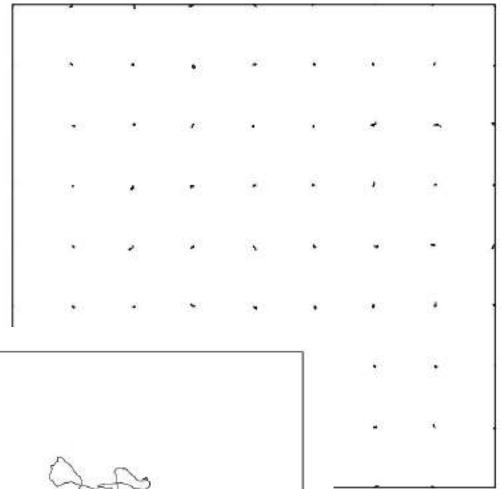
=

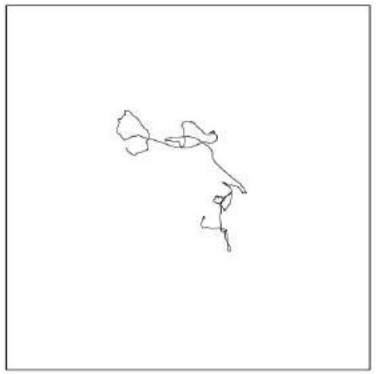
by the coherent flow



+

by the incoherent flow





Transport by the vortices

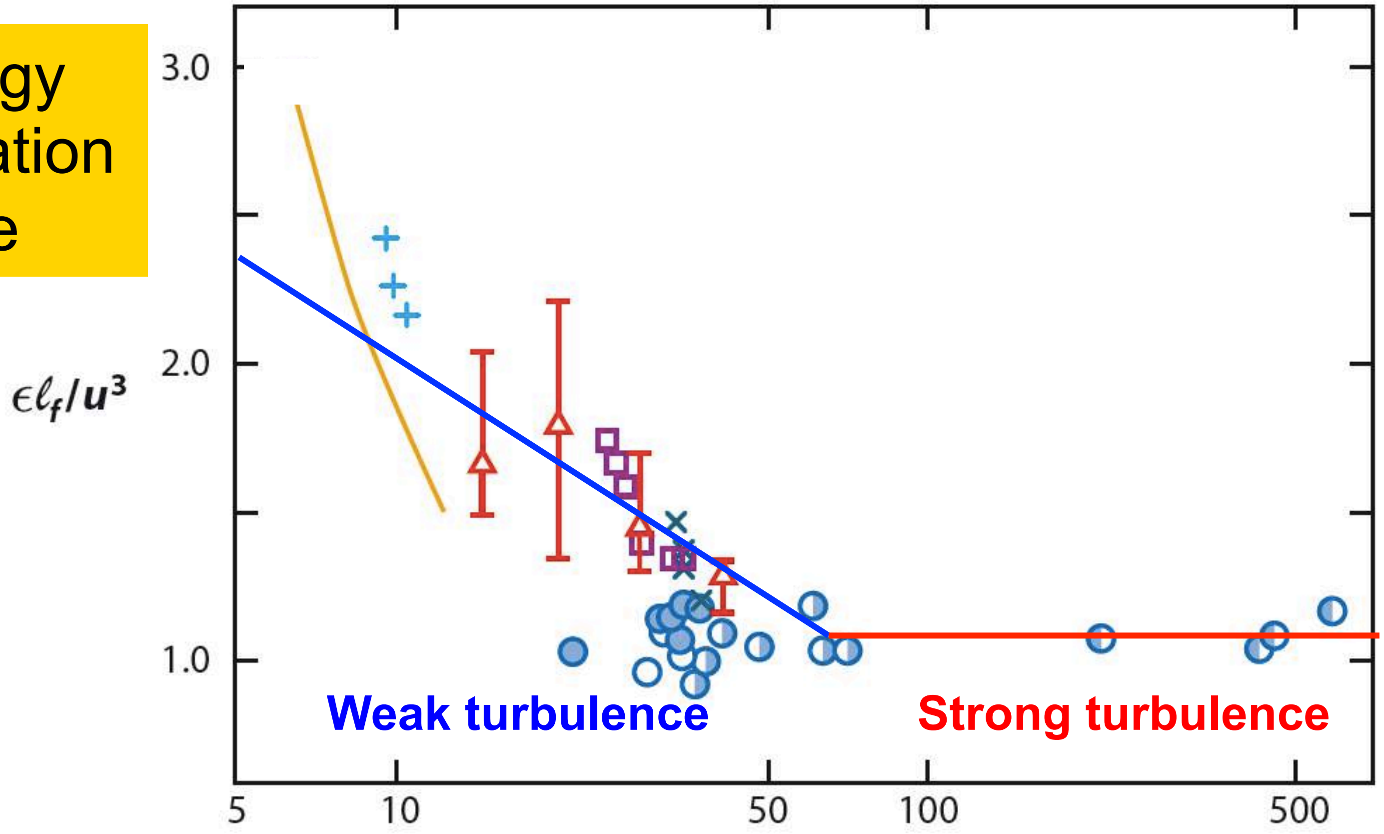
Diffusion by the noise as a Brownian motion

Beta,Schneider, M.F., 2003, Nonlinear Sci. Num. Simul., 8

Laboratory experiment of 3D turbulence

*Vassilicos, Ann. Rev. Fluid Mech., **47**, 2015*

Energy dissipation rate



$Re_\lambda = Re^{1/2}$

For $\nu \to 0$ or $\mathrm{Re} \to +\infty$

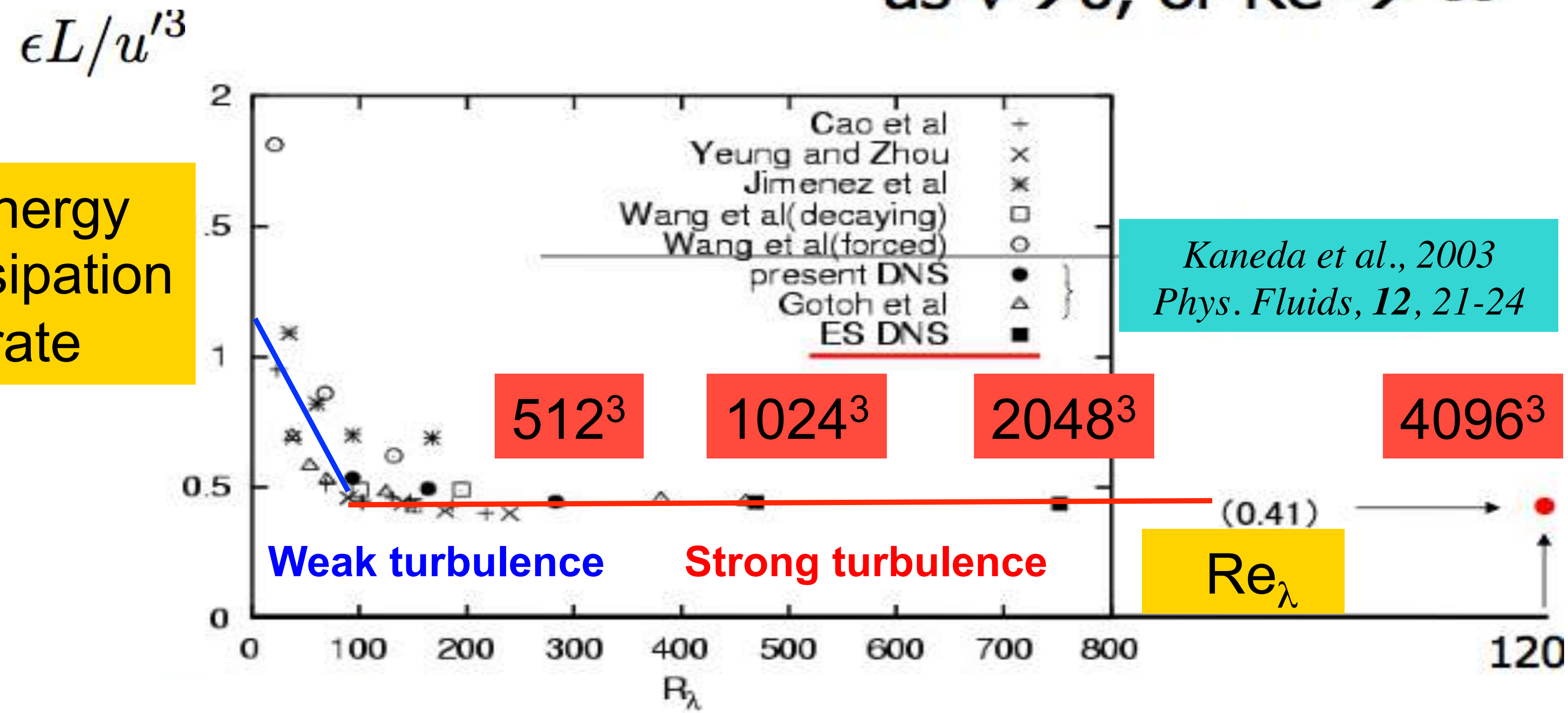
energy dissipation does not vanish but becomes constant

Numerical experiment of 3D turbulence

Normalized energy dissipation → ?

as $\nu \rightarrow 0$, or $Re \rightarrow \infty$

Energy dissipation rate



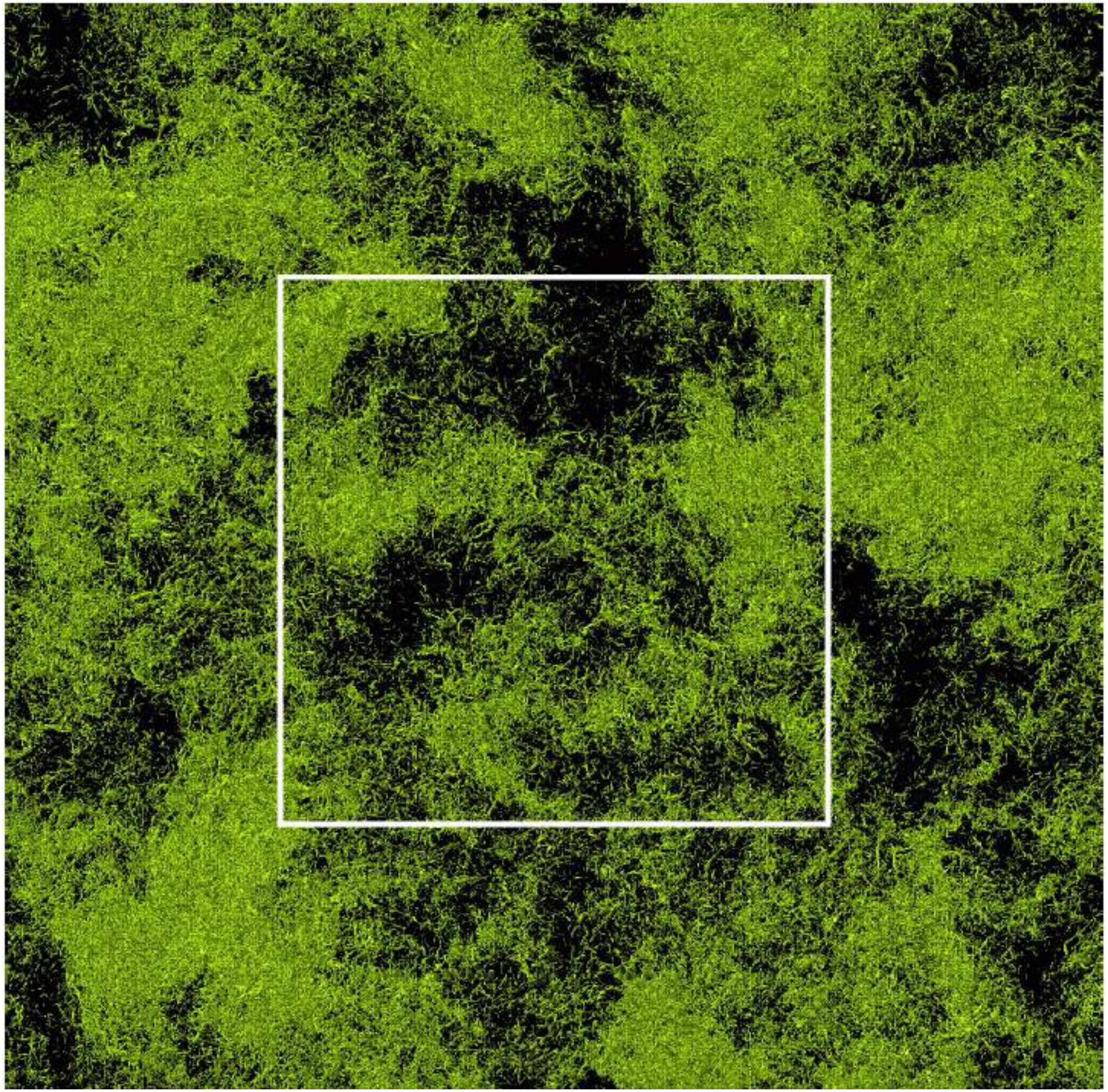
Both laboratory and numerical experiments show that the dissipation rate of turbulent flows becomes independent of the fluid viscosity for large *Re*

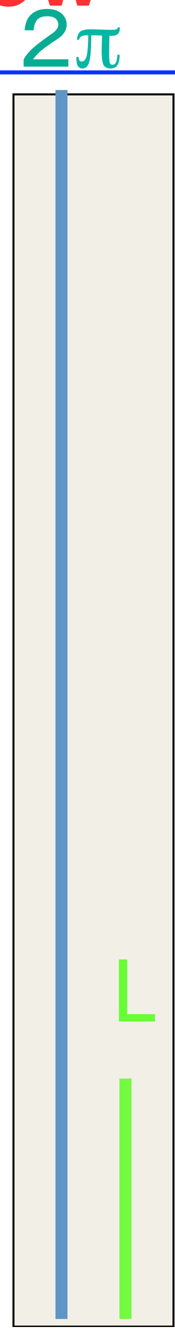
Wavelet filtering of a 3D turbulent flow

DNS
N=2048^3

L is the integral scale at which energy is injected

Okamoto, M.F., et al. 2007 Phys. Fluids, 19(11), 11519



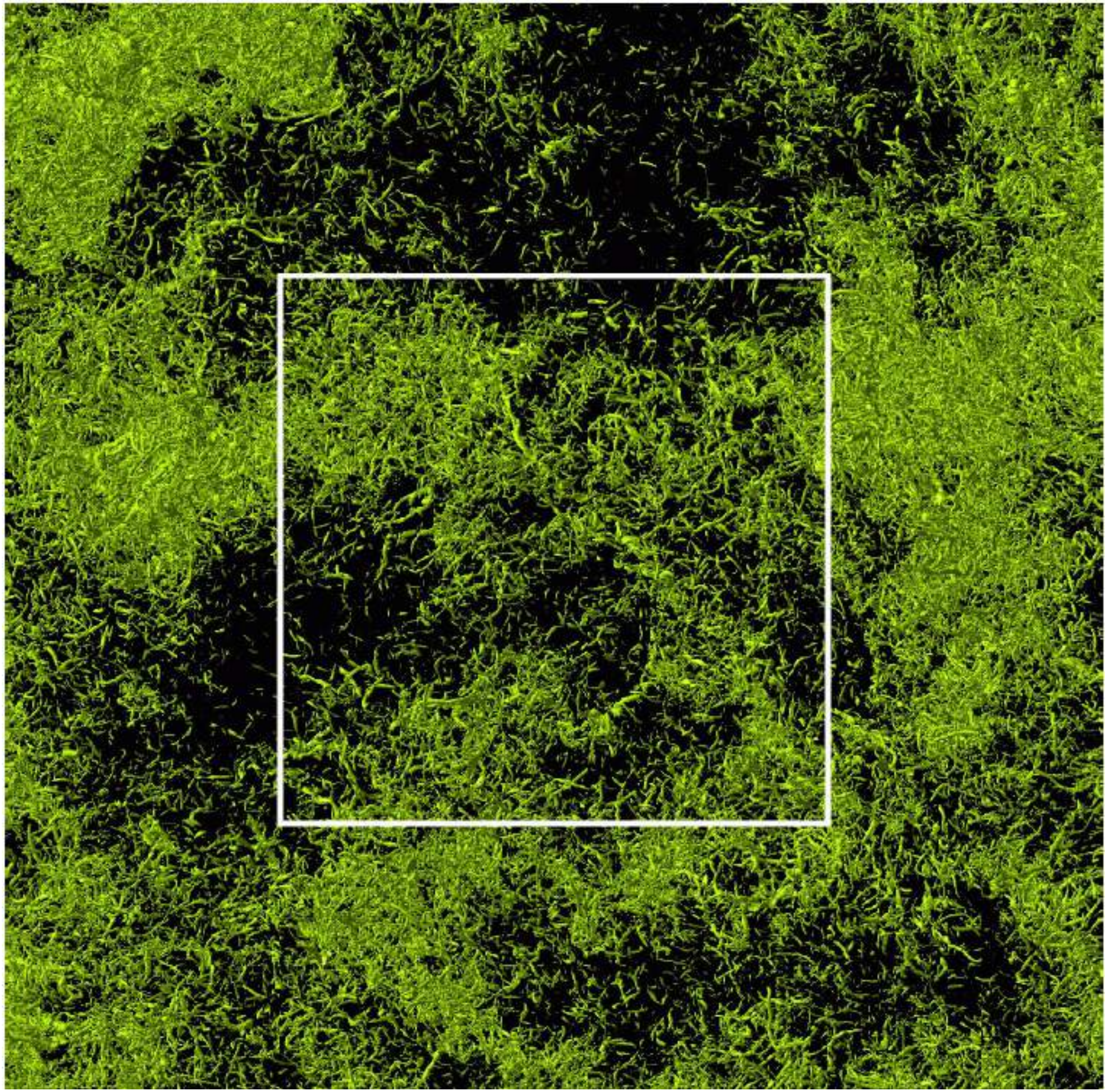


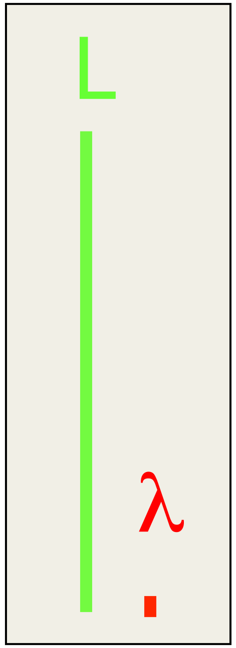
Zoom (sub-cube 1024^3)

Resolution
$N=2048^3$

L,
échelle
intégrale

λ,
micro-
échelle
de
Taylor





Okamoto, M.F., et al. 2007 Phys. Fluids, 19(11), 11519

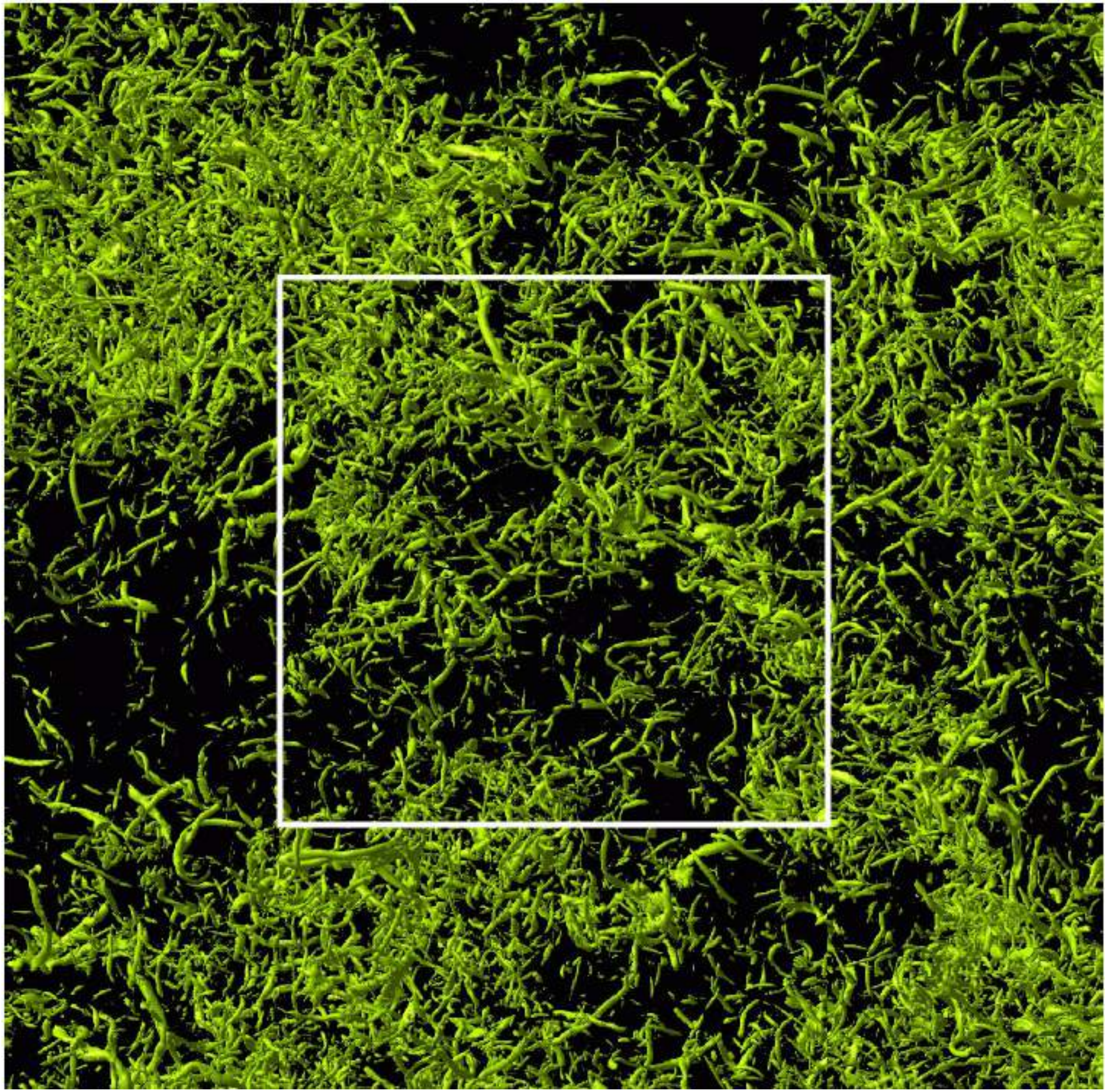
Zoom (sub-cube 512^3)

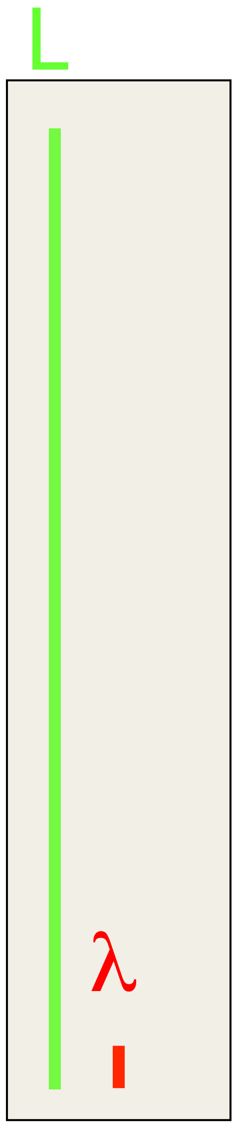
Resolution
N=2048^3

L,
échelle
intégrale

λ,
micro-
échelle
de
Taylor

Okamoto, M.F.,
et al. 2007
Phys. Fluids,
19(11), 11519





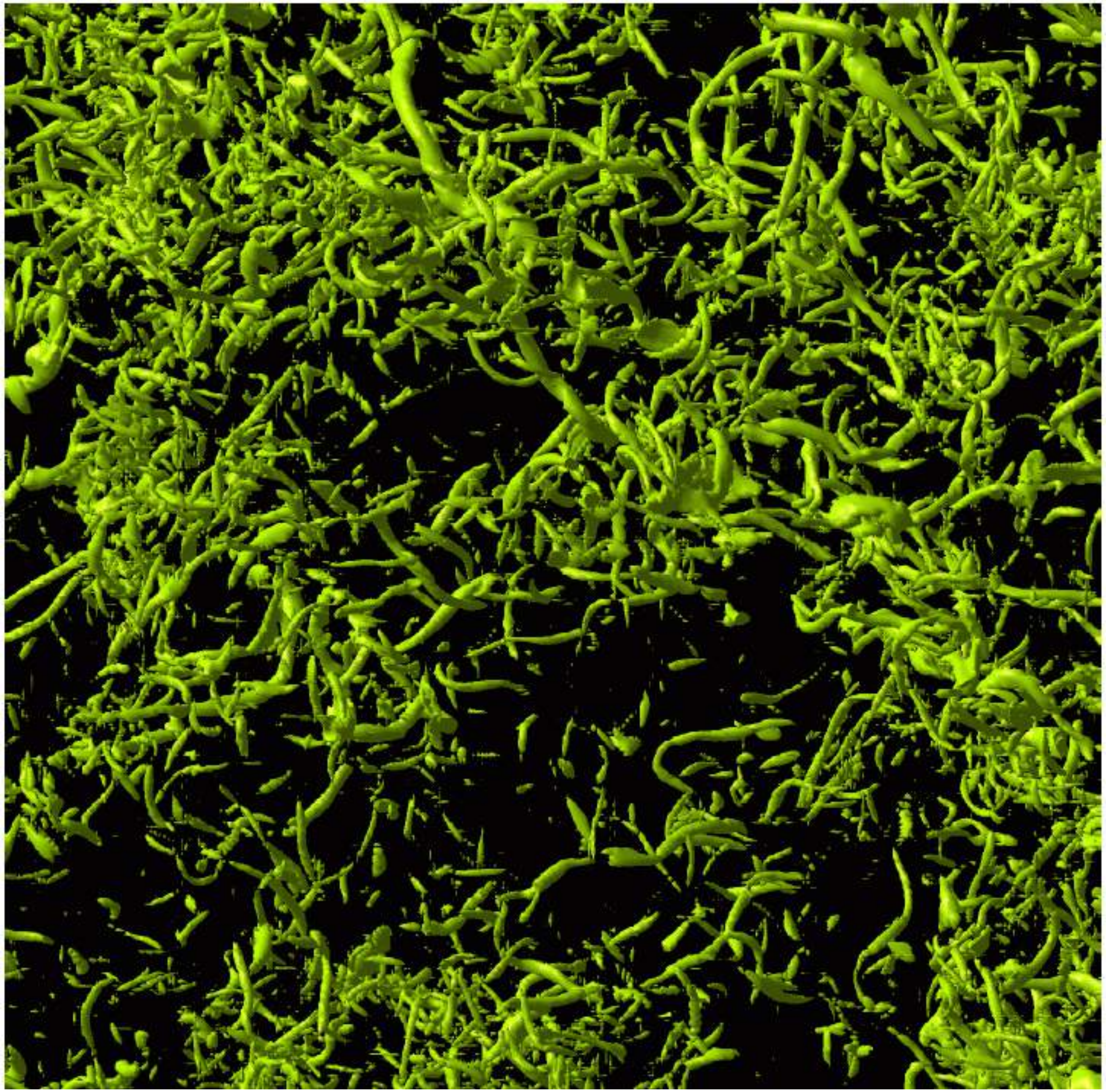
Zoom (sub-cube 256^3)

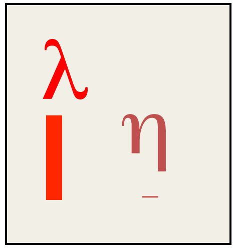
DNS
$N=2048^3$

λ,
micro-
échelle
de
Taylor

η,
échelle
dissipative
de
Kolmogorov

Okamoto, M.F., et al. 2007 Phys. Fluids, 19(11), 11519

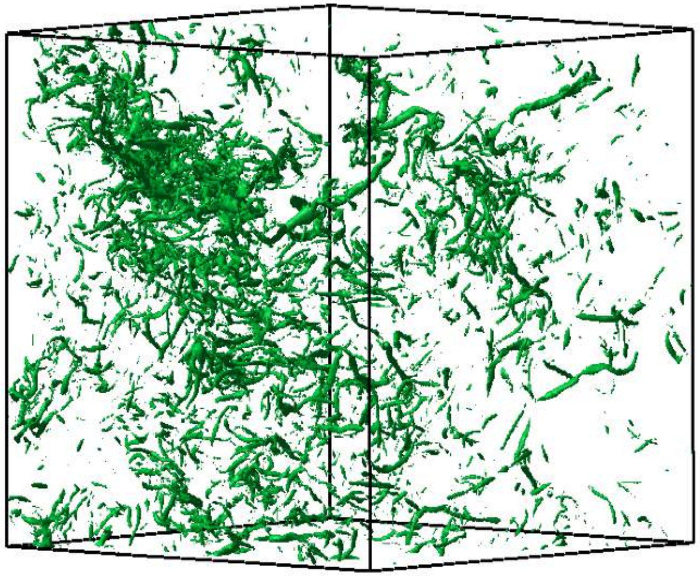




Wavelet filtering of a 3D turbulent flow

DNS
N=2048^3

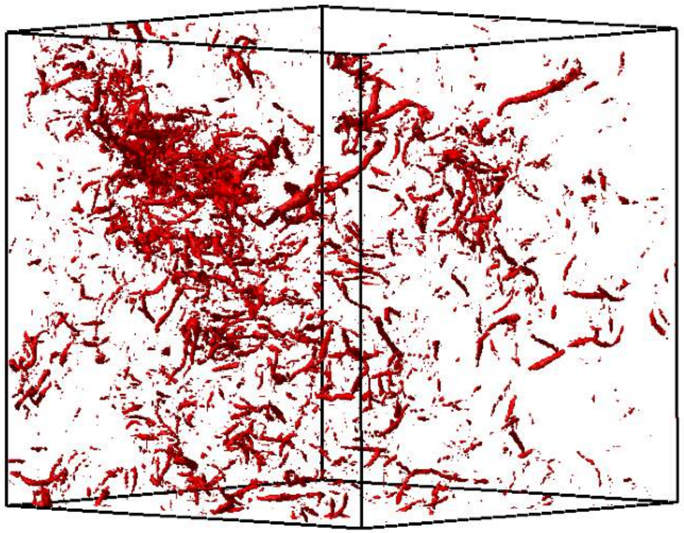
$|\omega|=5\sigma$ with $\sigma=(2Z)^{1/2}$



Total vorticity

Coherent vorticity

2.6 % N coefficients
80% enstrophy
99% energy

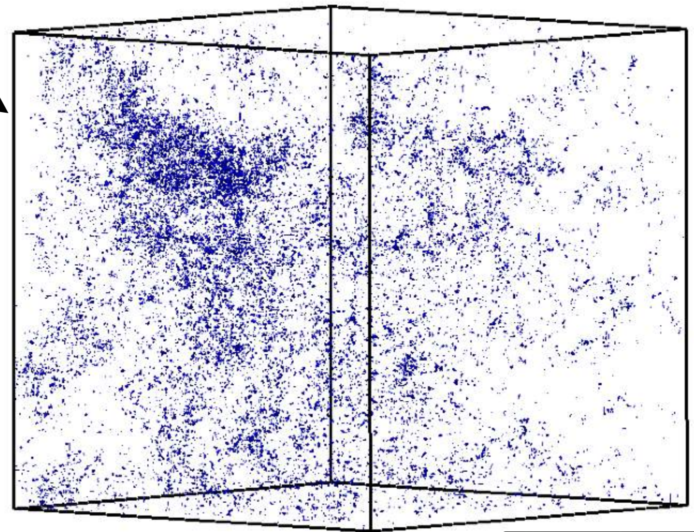


$|\omega|=5\sigma$

+

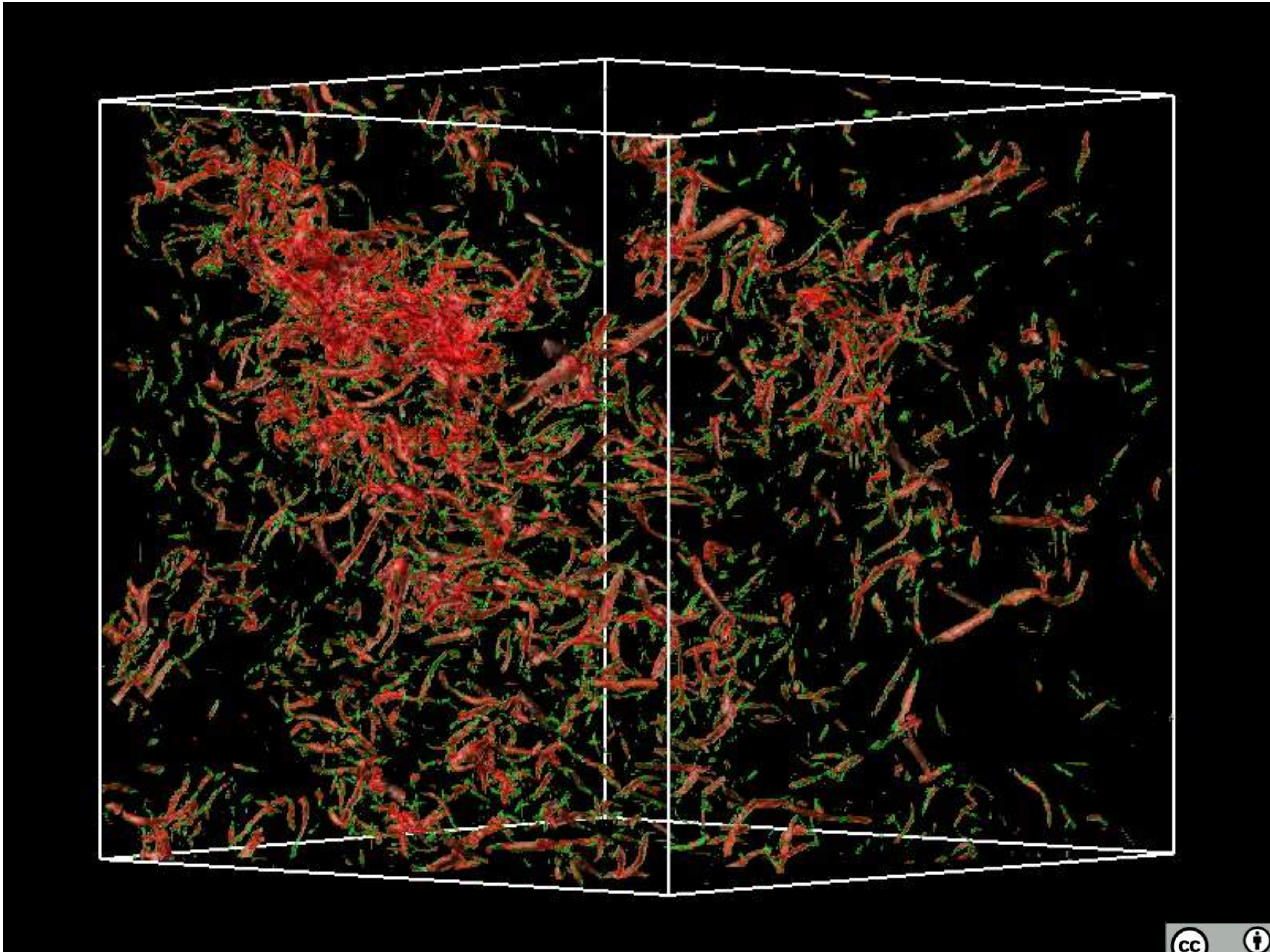
Incoherent vorticity

97.4 % N coefficients
20 % enstrophy
1% energy

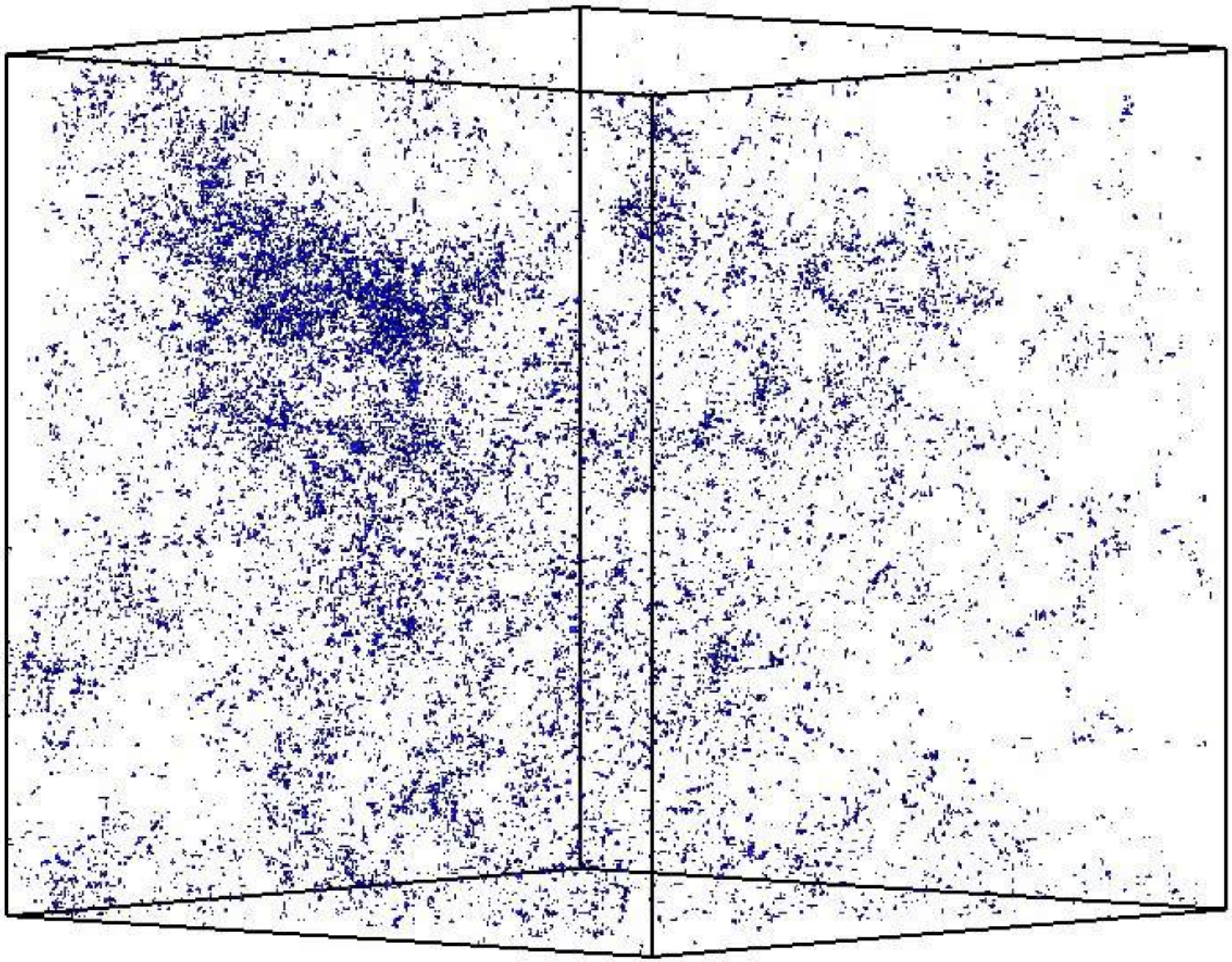


$|\omega|=5/3\sigma$

Okamoto, Yoshimatsu, Schneider, M.F., Kaneda, 2007, Phys. Fluids, ***19****, 1159*



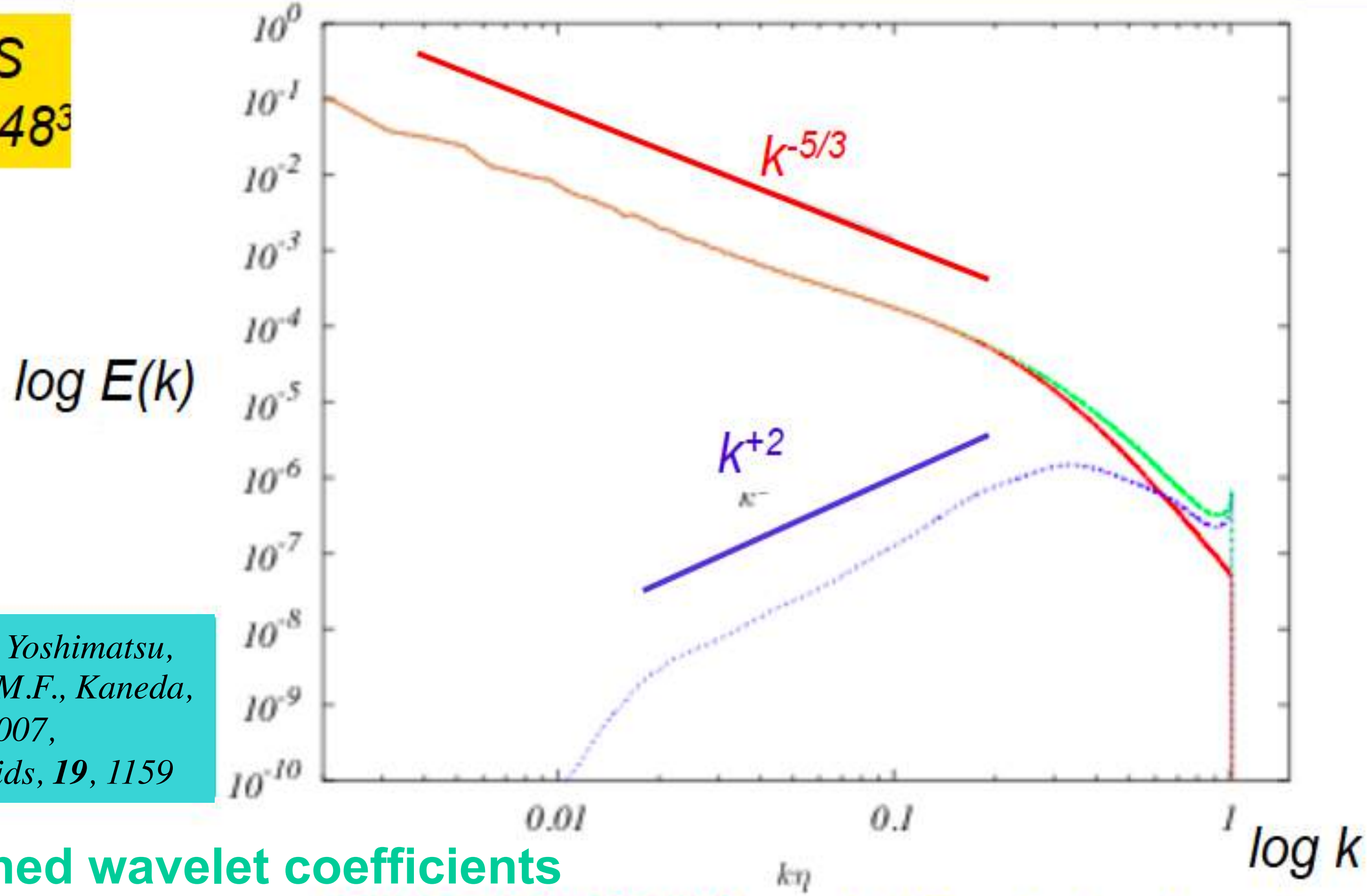
All vorticity tubes (green) have been extracted as coherent (red)



The remaining background flow does not contain vorticity tubes

Energy spectrum

DNS
$N=2048^3$



Okamoto, Yoshimatsu, Schneider, M.F., Kaneda, 2007, Phys. Fluids, ***19****, 1159*

# of retained wavelet coefficients

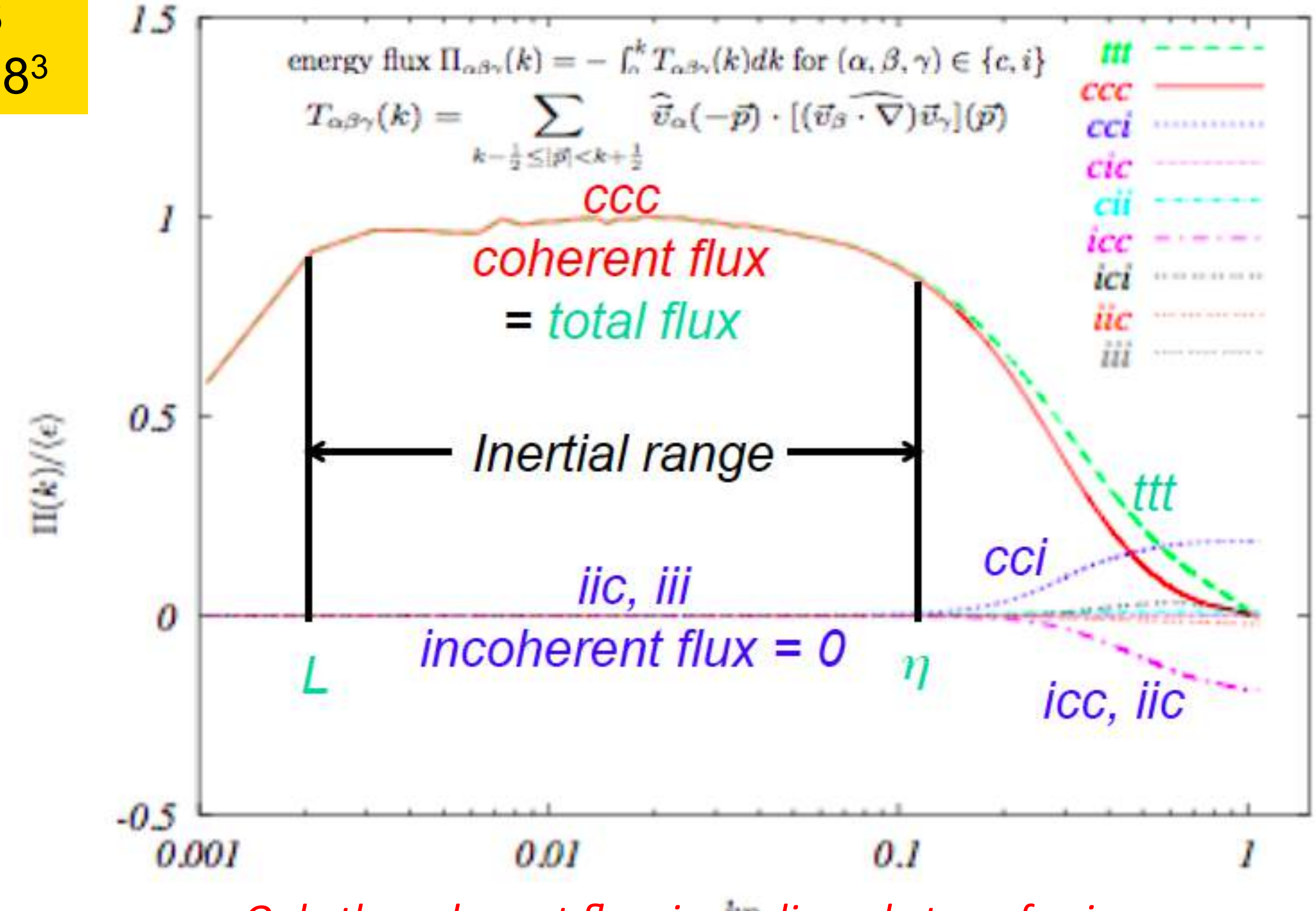
2.6 % N coefficients
80% enstrophy
99% energy

Multiscale Coherent
$k^{-5/3}$ scaling, i.e.
long-range correlation

Multiscale Incoherent
k^{+2} scaling, i.e.
energy equipartition

Nonlinear transfers and energy fluxes

DNS
N=2048^3



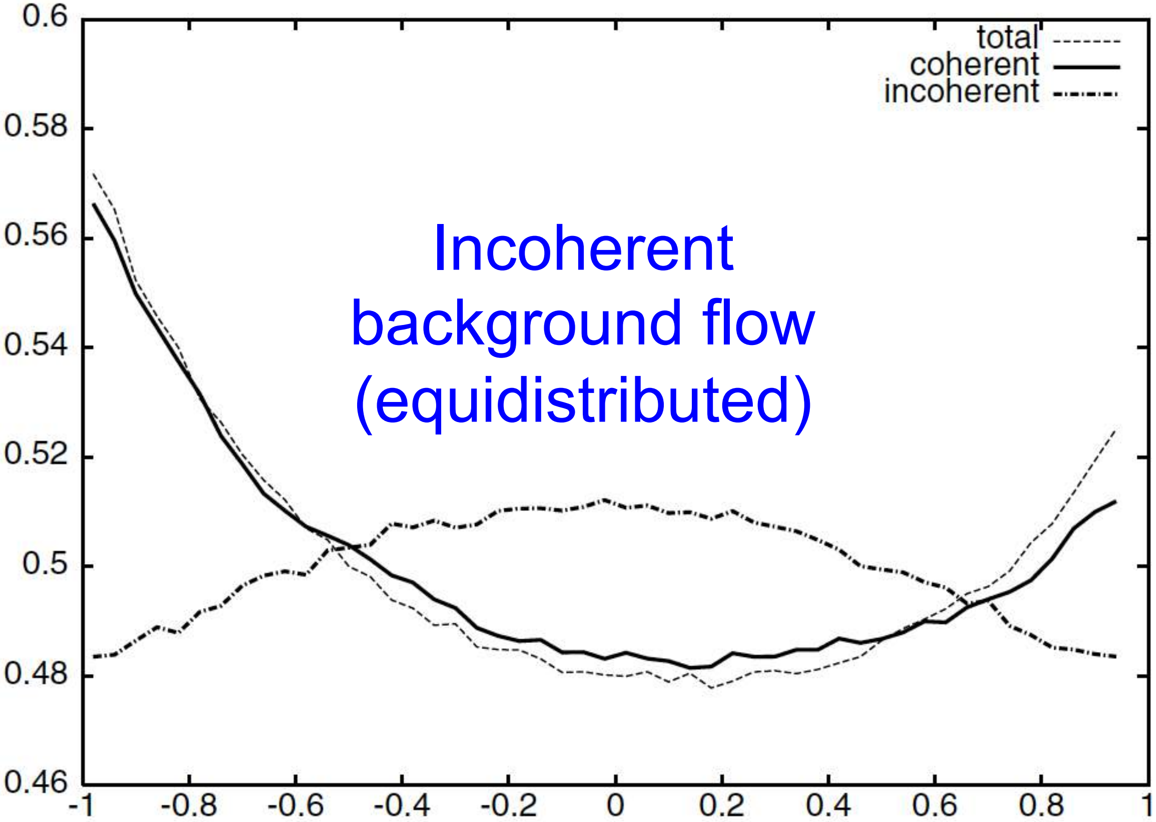
Only the coherent flow is nonlinearly transferring energy
while the incoherent flow is not active in the inertial range

Okamoto, Yoshimatsu, Schneider, M.F., Kaneda, 2007,
Phys. Fluids, ***19***, *1159*

PDF of relative helicity

Coherent vortex tubes with depletion of nonlinearity (peaked at |h|=1)

$$h = \frac{\vec{V} \cdot \vec{\omega}}{|\vec{V}||\vec{\omega}|}$$

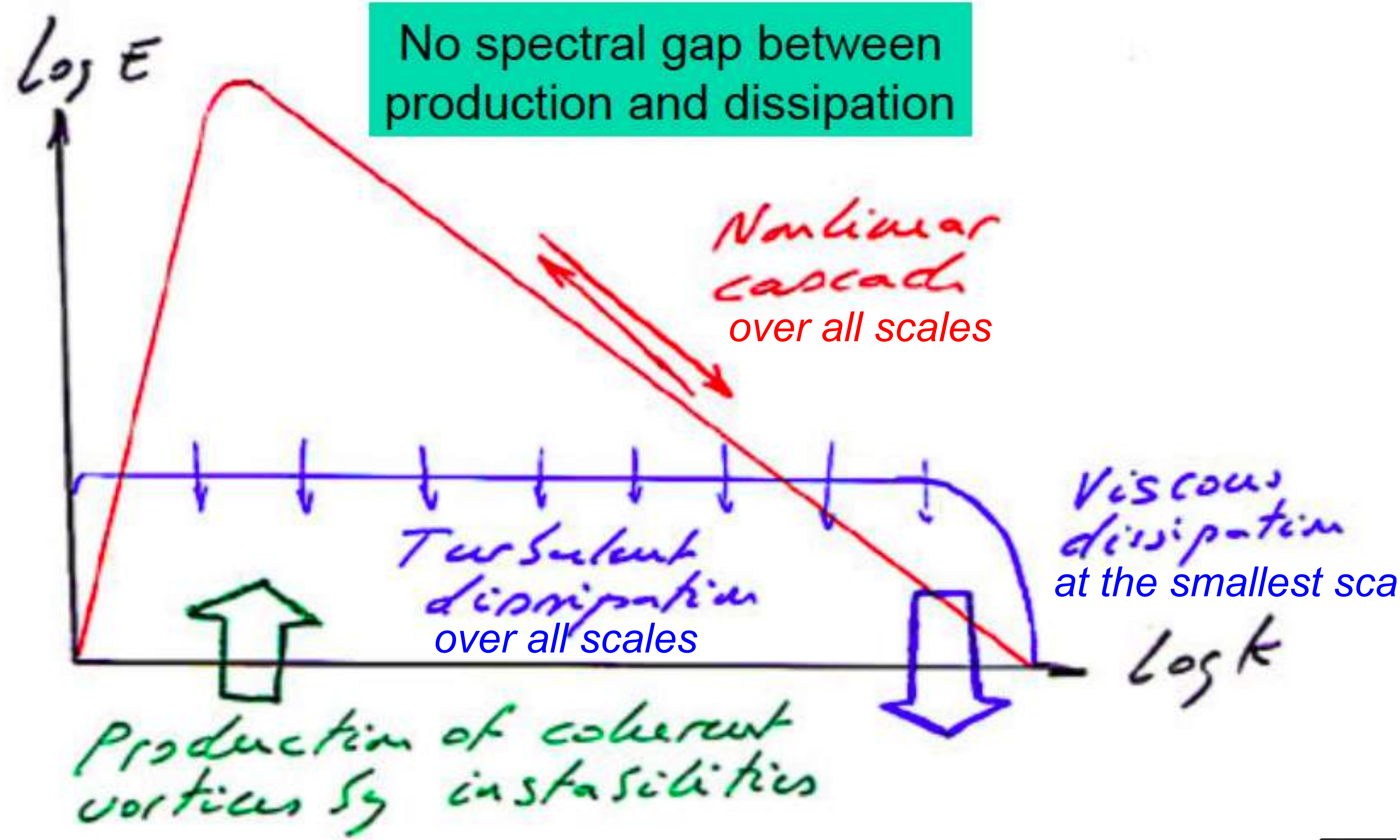


*M. F., Pellegrino and Schneider, 2001, 'Coherent vortex extraction in 3D turbulent flows using orthogonal wavelets', Phys. Rev. Lett.., **87**(5)*

HOW TO MODEL AND COMPUTE TURBULENT FLOWS ?

New interpretation of the turbulence cascade

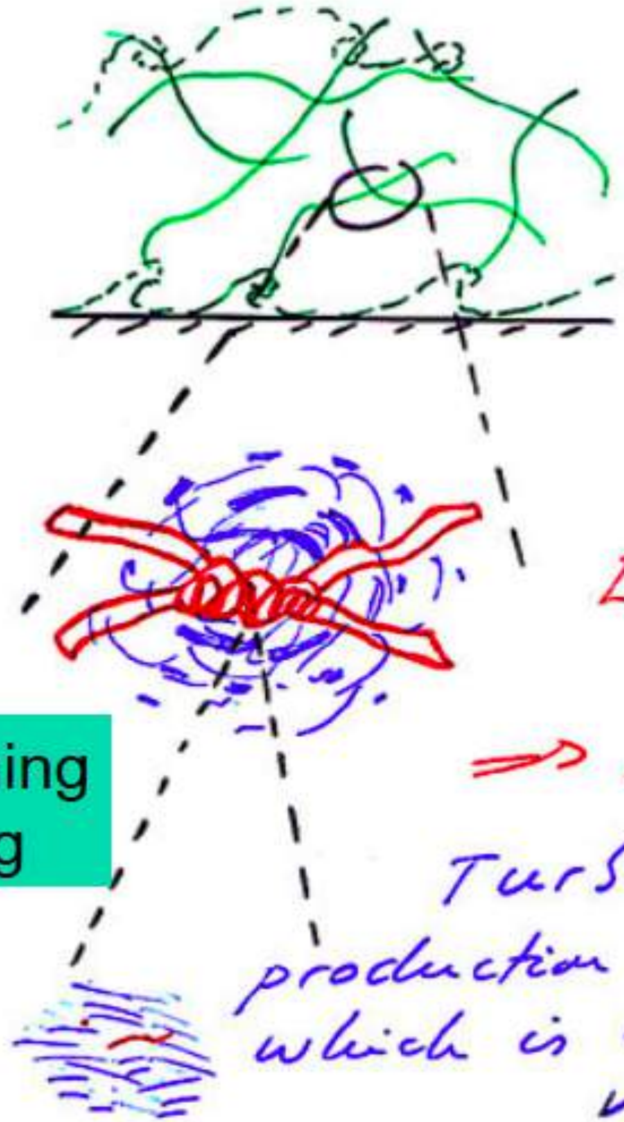
Fourier space viewpoint



New interpretation of the turbulence cascade

Physical space viewpoint

No vortex fission 'a la Richardson'



PRODUCTION

Webs of vortex tubes produced by instabilities in shear layers

INERTIAL RANGE

Intermittent vortex interactions

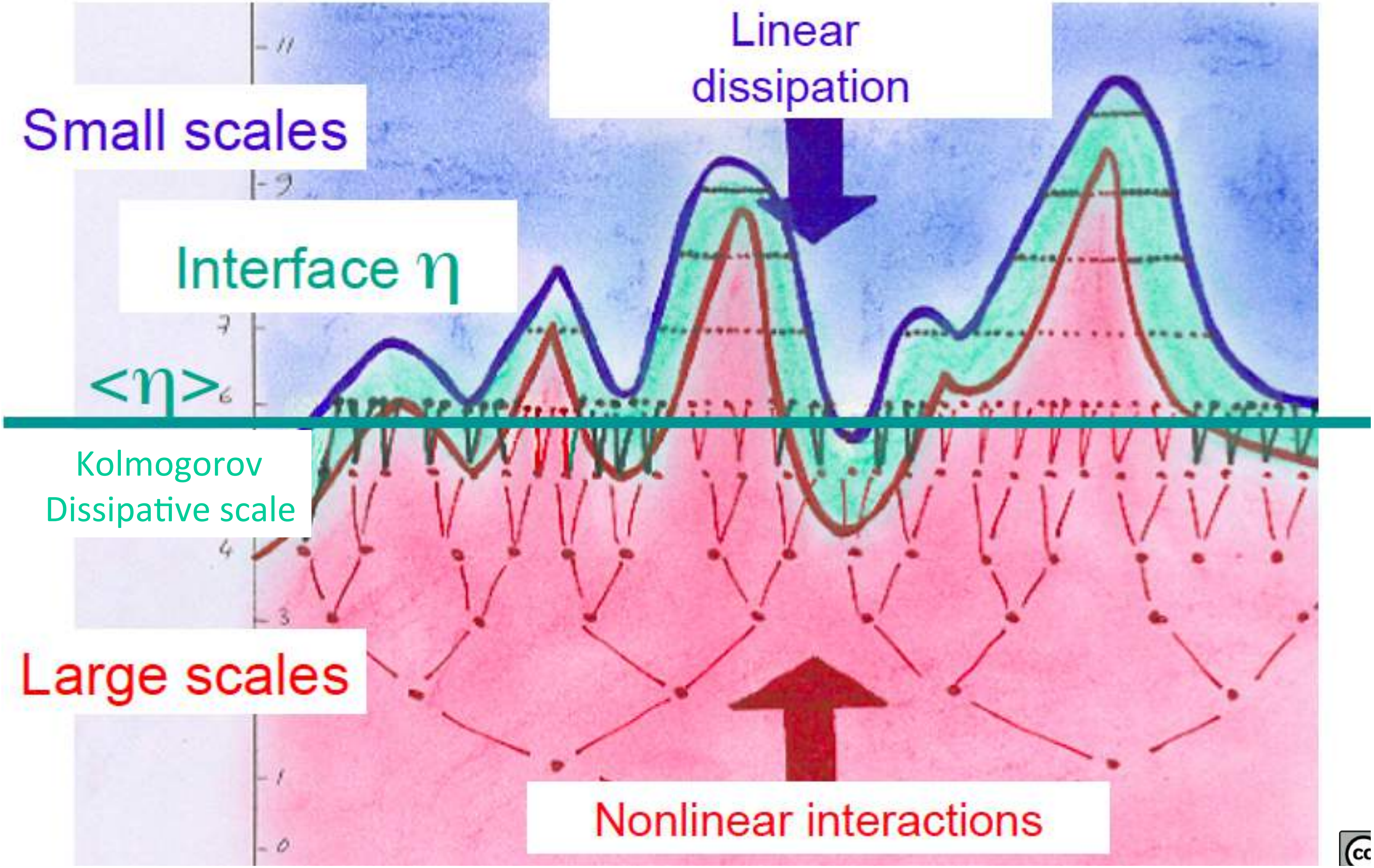
=> Nonlinear cascade + Turbulent dissipation

production of background noise which is damped in the viscous range

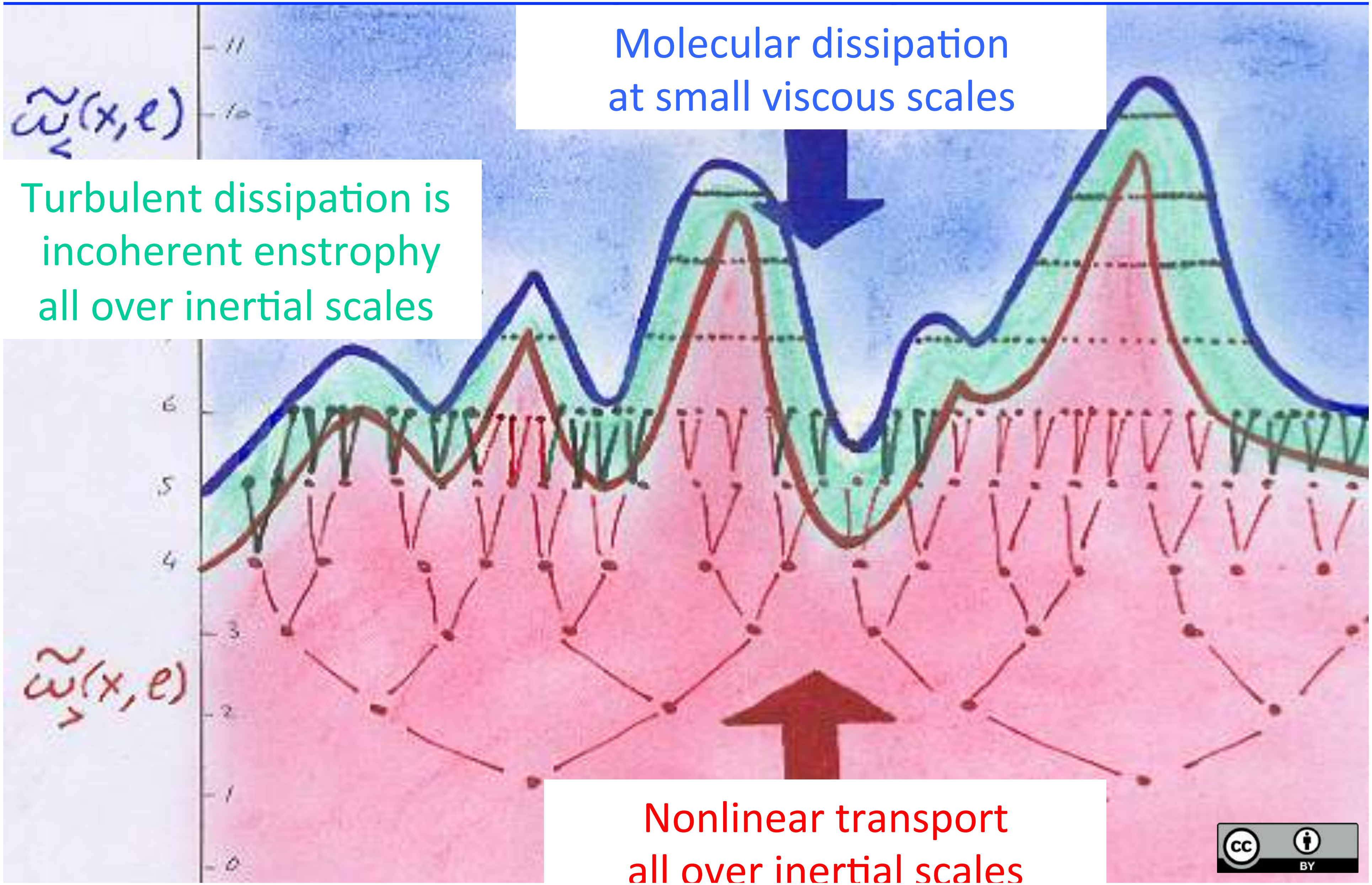
Vortex stretching and bursting

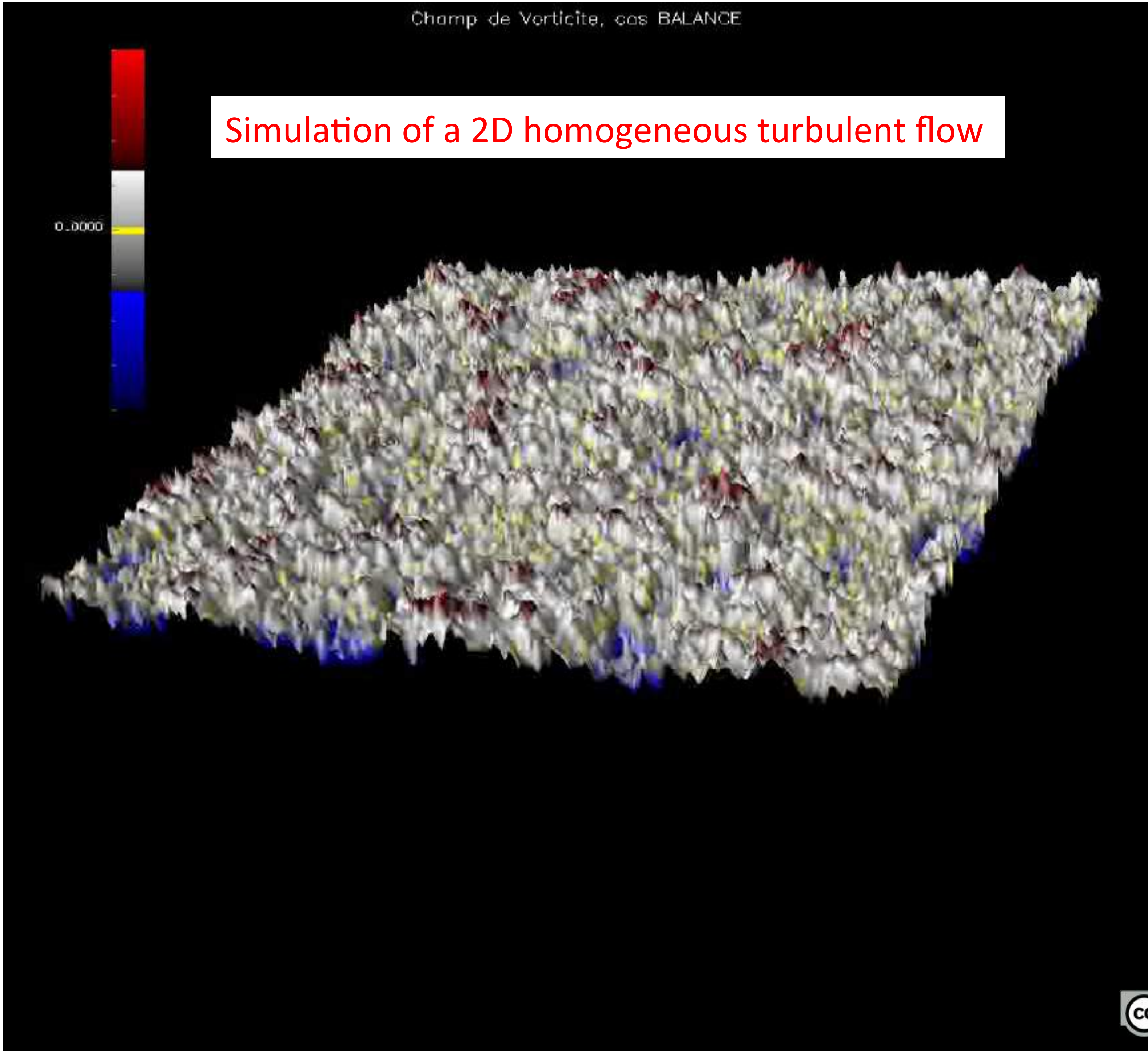
New interpretation of turbulence cascade

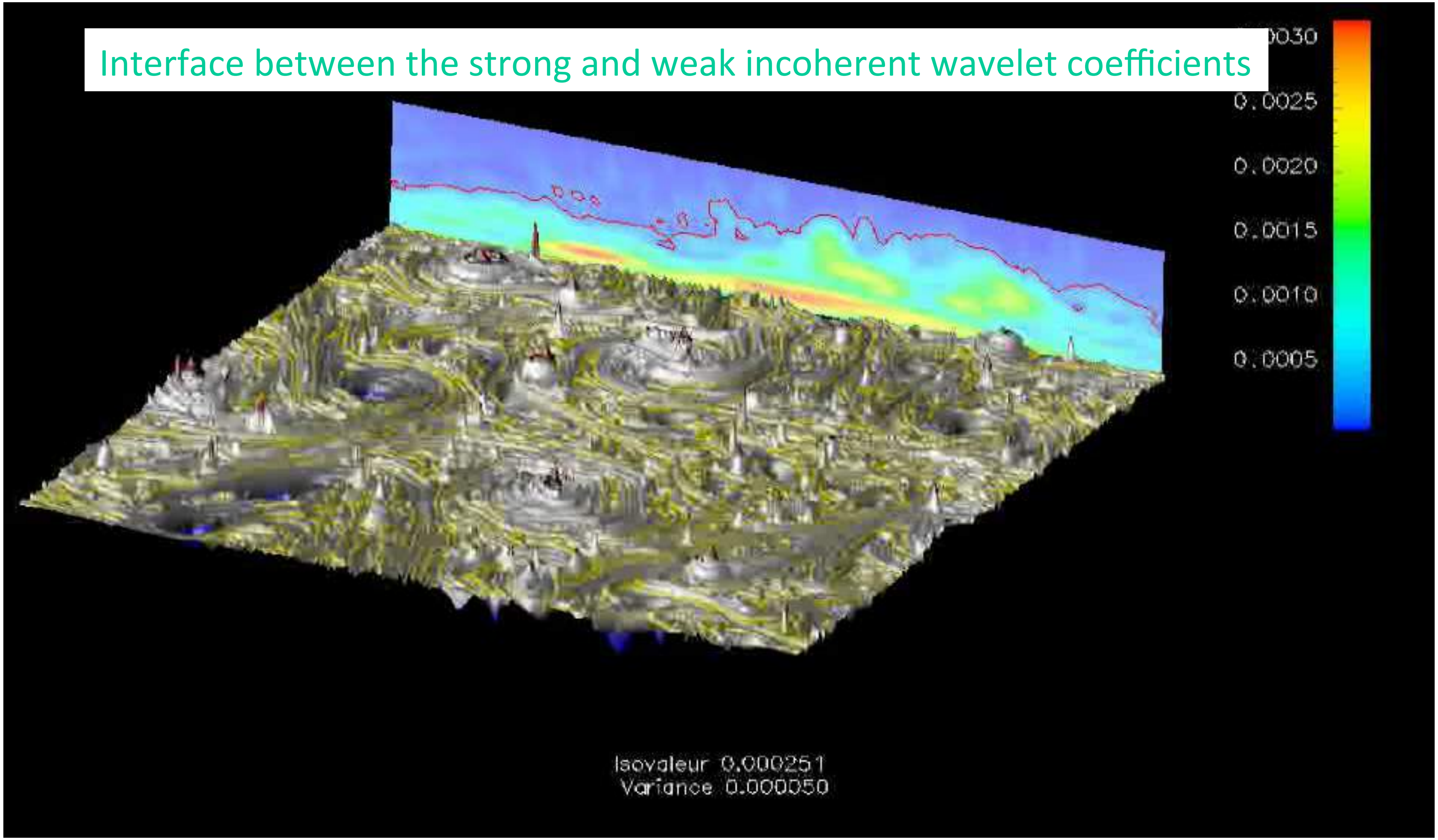
Wavelet space viewpoint



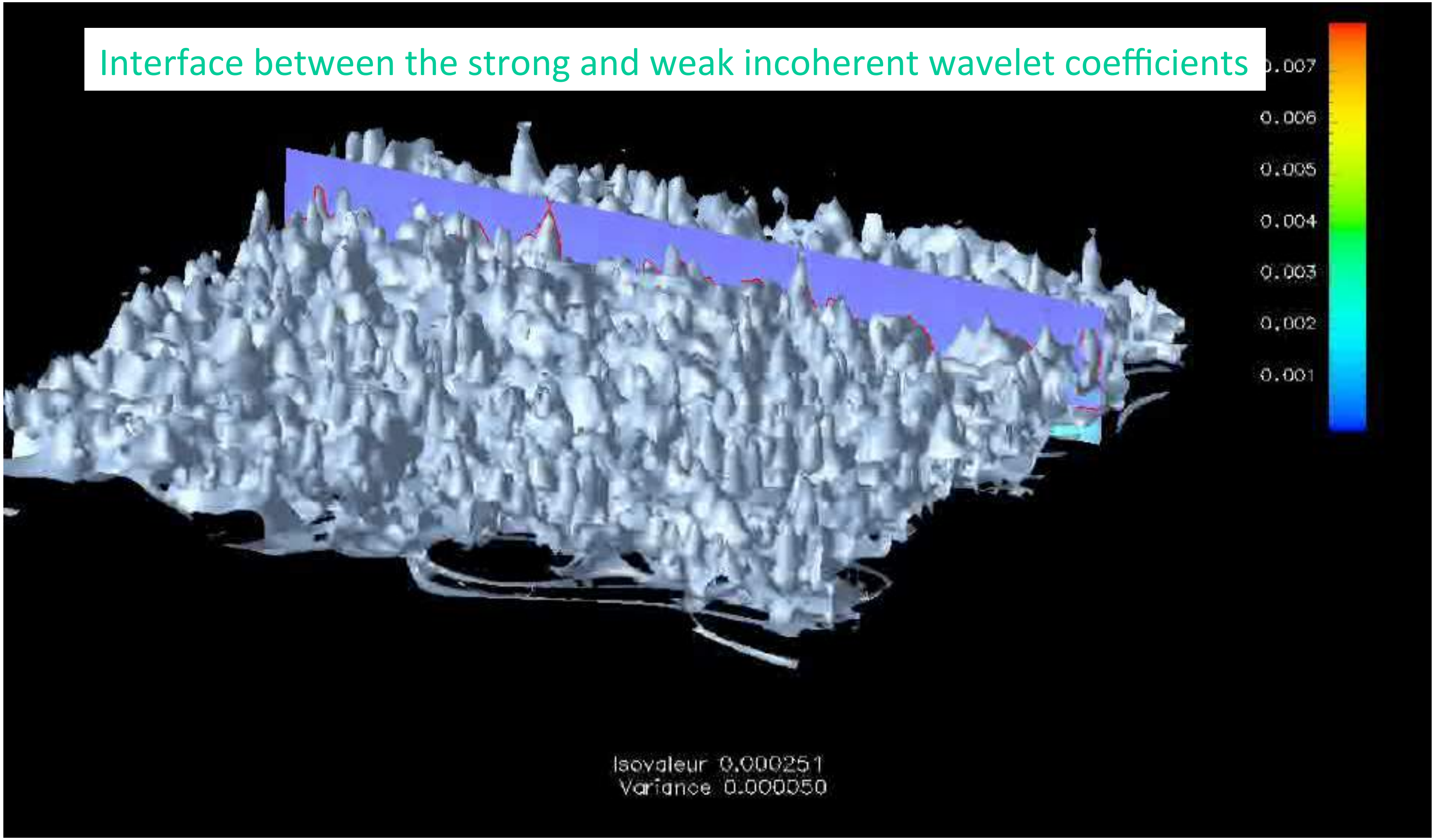
Wavelet-based definition of turbulent dissipation




Champ de Vorticite, cas BALANCE
Simulation of a 2D homogeneous turbulent flow
0.0000



The strong wavelet coefficients are below the interface
and correspond to the flow generated by the coherent vortices
The weak wavelet coefficients are above the interface
and correspond to the incoherent dissipative background flow



The strong wavelet coefficients are below the interface
and correspond to the flow generated by the coherent vortices.
The weak wavelet coefficients are above the interface
and correspond to the incoherent dissipative background flow.

'We conjecture that turbulent flows can be described as a superposition of metastable coherent vortices that are not in statistical equilibrium. Their nonlinear interactions are responsible for the chaotic behaviour of turbulent flows and generate a random incoherent flow, which then relaxes towards statistical equilibrium and is dissipated at the smallest scales.'

M. F., Pellegrino and Schneider, 2001,
'Coherent vortex extraction in 3D turbulent flows using orthogonal wavelets',
*Phys. Rev. Lett.., **87**(5)*

'We conjecture that the wavelet representation, formulated in terms of both space and scale, allows such a decoupling between organized motions out of statistical equilibrium and random motions in statistical equilibrium. Both components are multiscale but have different probability distributions and correlations.'

M. F., Pellegrino and Schneider, 2001,
'Coherent vortex extraction in 3D turbulent flows using orthogonal wavelets',
*Phys. Rev. Lett.., **87**(5), 2001*

‘This gives us incentives to extend the CVS method to compute three-dimensional Navier-Stokes equations in an adaptive wavelet basis, remapped at each time step to track the nonlinear vortex dynamics in both space and scale, as we have done for two-dimensional turbulent flows. The advantage of the CVS method is to combine an Eulerian representation of the solution in a wavelet basis with a Lagrangian strategy to adapt the basis in space and scale, to track the formation, advection, and dissipation of vortex tubes whatever their scales.’

M. F., Pellegrino and Schneider, 2001,
‘Coherent vortex extraction in 3D turbulent flows using orthogonal wavelets’,
*Phys. Rev. Lett.., **87**(5), 2001*

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- Dr. Thomas Engels, CR CNRS, ISM, Marseille, and TU Berlin

http://aifit.cfd.tu-berlin.de

M. F., 2022
The evolution of turbulence theories
and the need for continuous wavelets
arXiv: 2209.01808

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13 January 2017 at 9 50 a.m.

M. F., 1992
Wavelet transforms and their applications to turbulence,
*Ann.Rev.Fluid Mech., **24**, 395-457*

http://turbulence.ens.fr

http://openscience.ens.fr/MARIE_FARGE